Lecture 11
Multi-Rate Signal Processing and Filterbanks

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Contents/Lecture 11

Decimation and interpolation

- Decimation as discrete-time subsampling
- Interpolation as discrete-time signal reconstruction: zero insertion and interpolation filtering
- Block diagram and z-transform notation
- Example: trans-multiplexer

Abtastraten Reduktion:

$$\begin{tikzcd} x[n] \arrow[r] & \boxed{\downarrow M} \arrow[r] & \tilde{x}[n] \end{tikzcd}$$

$x[n] \neq \tilde{x}[n] \Rightarrow x[n] = \tilde{x}[n \cdot M]$

↳ reduziert #Samples um Faktor M

$$\begin{tikzcd} x[n] \arrow[r] & \boxed{z^{-1}} \arrow[r] & \boxed{\downarrow M} \arrow[r] & \tilde{x}[n] \end{tikzcd}$$

$\tilde{x}[n] = x[n \cdot M - 1]$

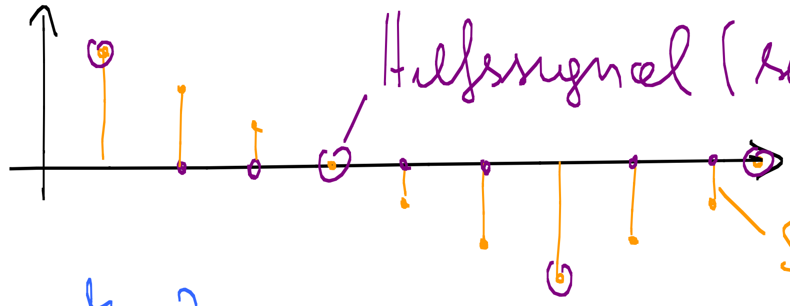
$$\begin{tikzcd} x[n] \arrow[r] & \boxed{\downarrow M} \arrow[r] & \boxed{z^{-1}} \arrow[r] & \tilde{x}[n] \end{tikzcd}$$

$\tilde{x}[n] = x[nM - 1 \cdot M] = x[(n-1)M]$

Achtung: $\boxed{\downarrow M}$ ist nicht zeitinvariant
(es verschieben sich verschiebungen gibt:
am Eingang um M Stellen schieben $\Rightarrow$ 1 Stelle
am Ausgang

das System ist jedoch linear

Analyse im Frequenzbereich:



Hilfssignal (selbe Werte wie Reduktion, jedoch selbe Samplingrate wie Signal)

Signal

$M = 3$

6

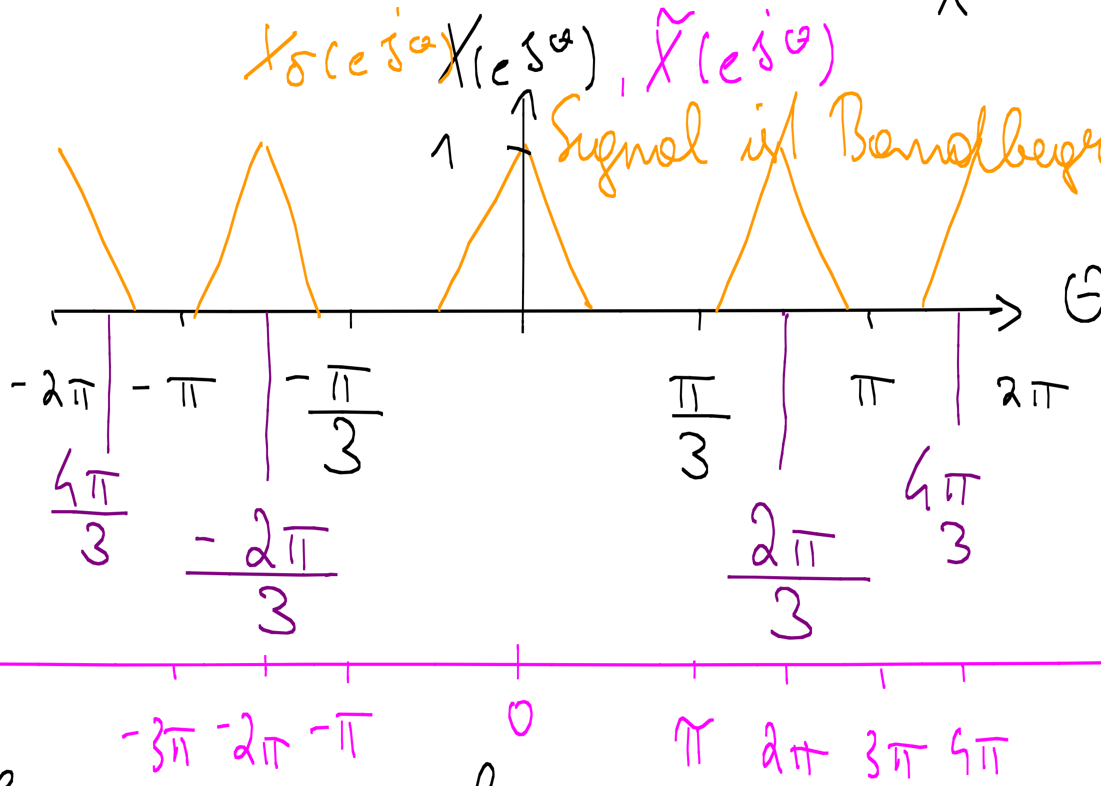
$$
\begin{array}{ccccc}
x & \rightsquigarrow & x_\delta & \rightarrow & \tilde{x} \\
\circ\!\!-\!\!\bullet & & \circ\!\!-\!\!\bullet & & \circ\!\!-\!\!\bullet \\
X & & X_\delta & & \tilde{X}
\end{array}
$$

→ setzt jeden M-ten Wert auf den Wert von x

$$x_\delta[n] = \sum_{k=-\infty}^{\infty} \underset{n=6}{x[n]}\; \underset{n=6 \quad k=2}{\delta[n - k \cdot M]} = x[n] \cdot \sum_k \delta[n - k \cdot M]$$

$$M \cdot \sum_k \delta[n - kM] \circ\!\!-\!\!\bullet\; 2\pi \sum_l \delta\left(\Theta - l \frac{2\pi}{M}\right)$$

$$X_\delta(e^{j\Theta}) = \frac{1}{2\pi} X(e^{j\Theta}) * \left(2\pi \sum_\ell \delta\left(\Theta - \ell \underbrace{\frac{2\pi}{M}}_{\text{Periode } \frac{2\pi}{3}}\right)\right)$$



downsampling ergibt ein neues Spektrum

$\tilde{\Theta} = \Theta \cdot M$ ①

$\Rightarrow$ periodische Fortsetzung führt zu keinem Aliasing

$$\tilde{X}(e^{j\tilde{\Theta}}) = \sum_{k=-\infty}^{\infty} \tilde{x}[k] e^{-j\tilde{\Theta}k} = \sum_k x_\delta[M \cdot k] e^{-j\tilde{\Theta}k}$$

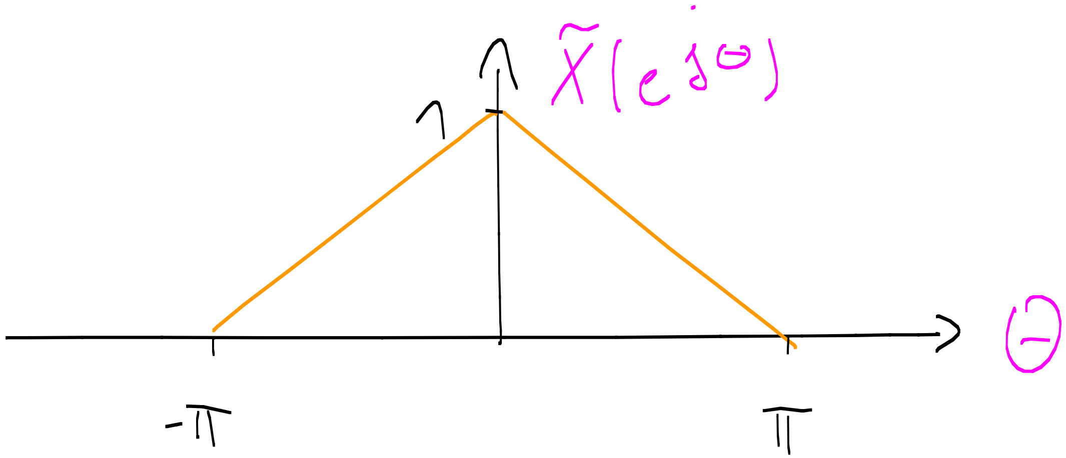
$$= \sum_n x_\delta[n] e^{-j\tilde{\Theta}\frac{k}{M}}$$

zählt man Oer hinzu ändert das nichts an einer Summe

ist $\frac{k}{M} \in \mathbb{R}$ ist egal, da in solchen Fällen $x_\delta[n] = 0$

Fourier-Trans ist eine Summe!

$$= \sum_n x_\delta[n] e^{-j\Theta n} \Big|_{\Theta = \frac{\tilde{\Theta}}{M}} = X_\delta(e^{j\Theta}) \Big|_{\Theta = \frac{\tilde{\Theta}}{M}}$$ ① neues Spektrum



$$X(e^{j\Theta}) = X(e^{j\frac{\Theta}{M}})$$

$$\tilde{X}(z) = X(z^{\frac{1}{M}})$$ M Werte weg lassen bedeutet $\sqrt[M]{}$ am Einheitskreis

$$\rightarrow \boxed{z^{-1}} \rightarrow \boxed{\downarrow M} \rightarrow \quad \Longleftrightarrow \quad \rightarrow \boxed{\downarrow M} \rightarrow \boxed{z^{-1}} \rightarrow$$

beide sind bezüglich der Übertragungsfunktion äquivalent

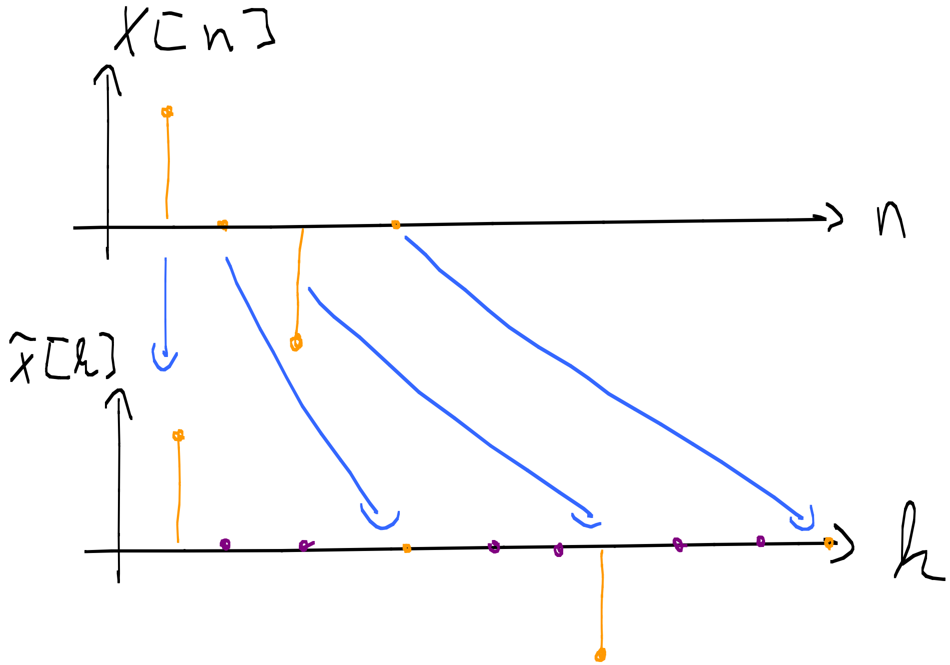
z.B.: Rechenoperationen erst nach einem downsampling durchführen um Ressourcen zu sparen.

Abtastratenerhöhung (engl. upsampling)

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \tilde{x}[n]$$

↳ erhöht #Samples um Faktor L

$$\tilde{x}[n] = \underbrace{\begin{cases} x[n] & , \; k = L \cdot n \\ 0 & , \; k \neq L \cdot n \end{cases}}_{\text{Es muss mit 0er aufgefüllt werden;}}$$

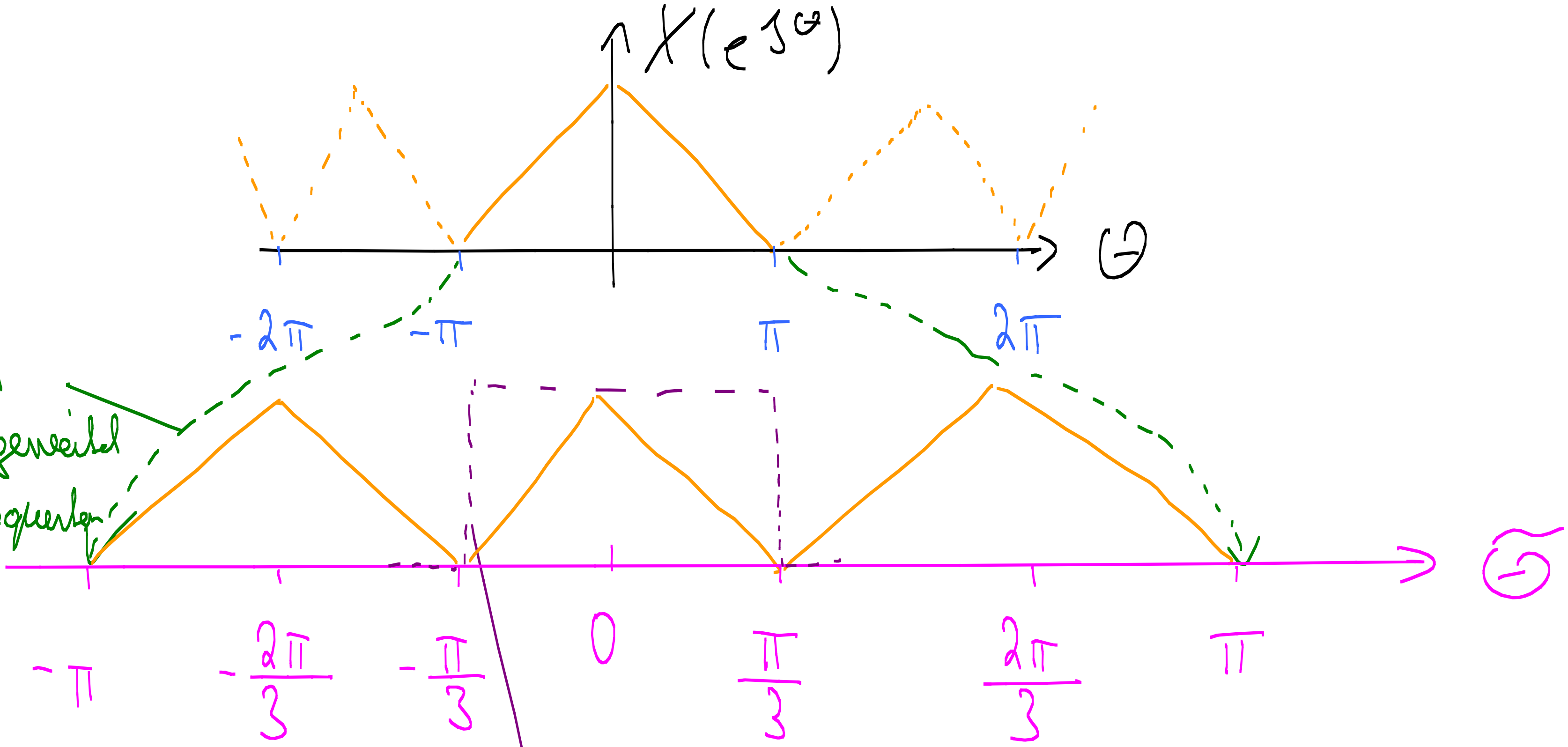


z.B.: $L = 3$

Upsamplen erzeugt den höheren Takt, jedoch keine sinnvollen Werte die ev. als Artefakte in Bildern erkannt werden

⇒ Interpolation verwenden

$\Theta \to \tilde{\Theta} = \Theta \cdot \frac{1}{L}, \quad z^{-1} \to z^{-L}$



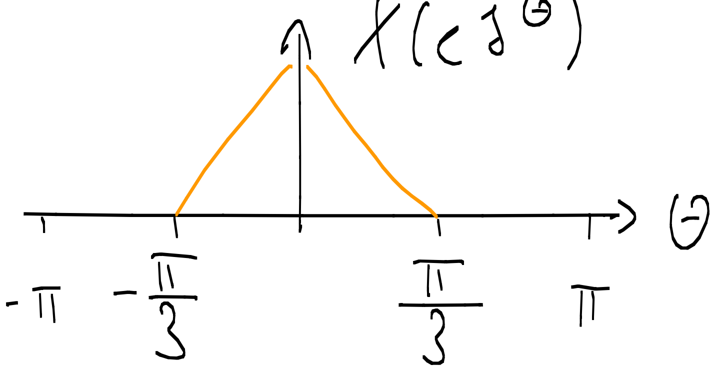
Spektrum wird aufgeweitet mit hoch-frequenten Anteilen
$\Rightarrow$ störend

Tiefpass mit $\omega_c = \frac{\pi}{3}$ um die hoch-frequenten Signale weg zu filtern ($\approx$ Interpolation)

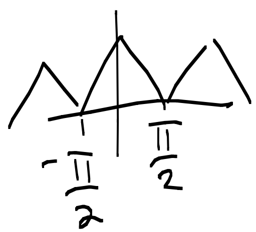
Interpolation:

$\Theta_c = \frac{\pi}{L}$

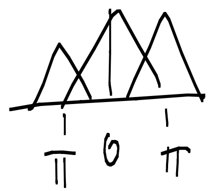
$x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{\text{interpolierendes Tiefpassfilter}} \rightarrow \tilde{x}[n]$



Beim downsampling besteht die Gefahr des destruktiven Alliasings z.B.:



$\rightarrow \downarrow M = 3$

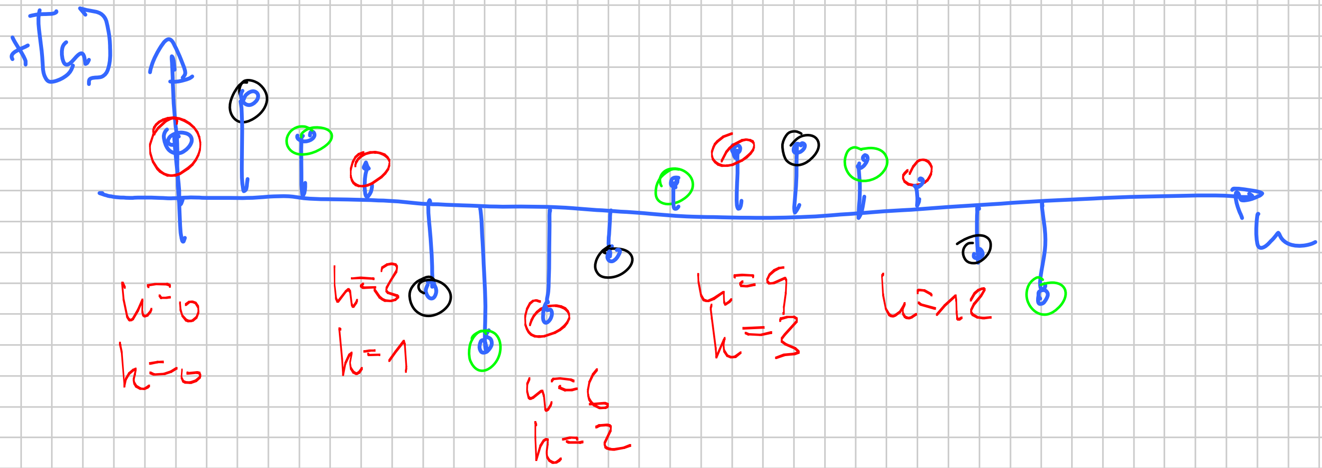


daher muss durch eine Bandbegrenzung (genannt Dezimations-filter) dies verhindert werden (meist auch als Tiefpass).

$x[n] \rightarrow \boxed{\text{Dezimationsfilter (TP)}} \xrightarrow{\Theta_c = \frac{\pi}{M}} \boxed{\downarrow M} \rightarrow \tilde{x}[n]$

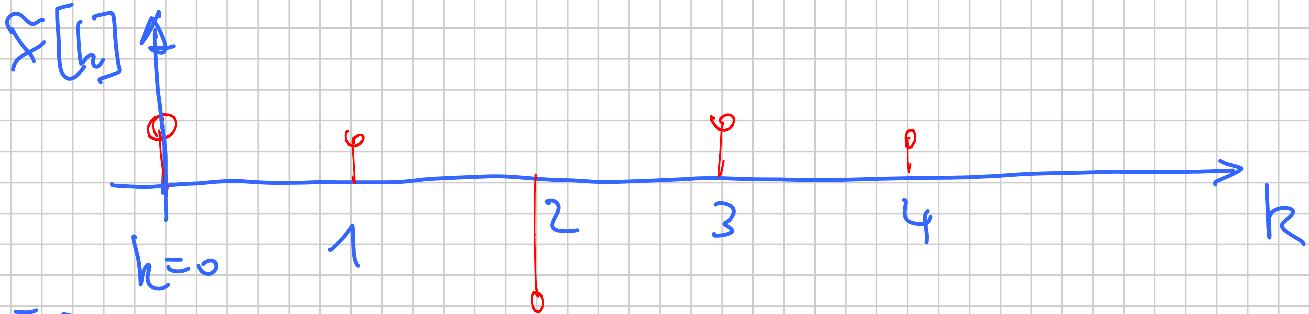
Filter errechnet Werte die beim downsamplen verworfen werden!

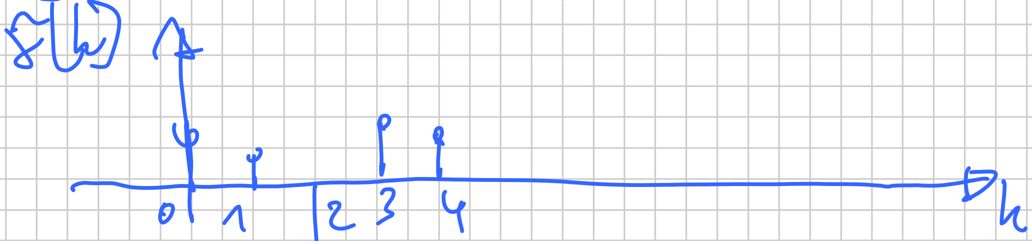
Unterabtastung / Taktratenreduktion



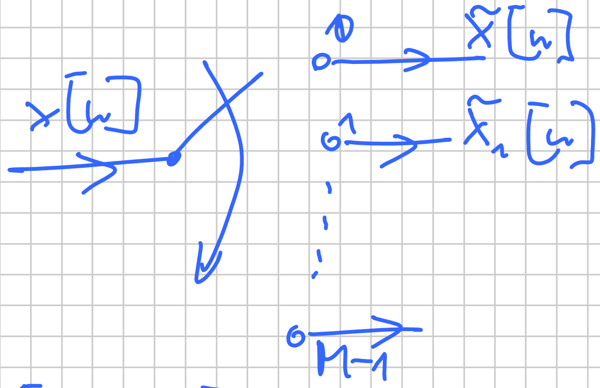
$$n = k \cdot M$$

$$\boxed{M=3}$$





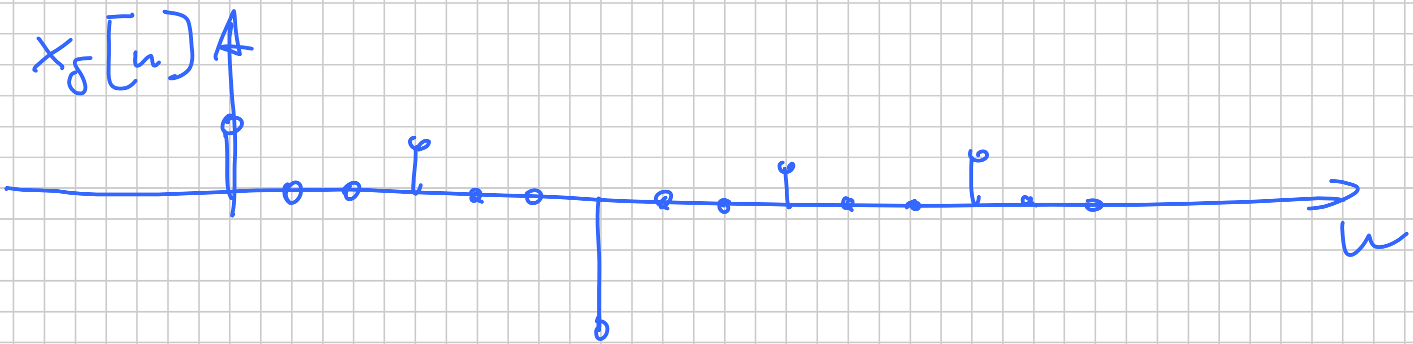
$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow \tilde{x}[n]$$



$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow \boxed{\downarrow M} \longrightarrow \tilde{x}_1[n] = x[nM-1]$$

$$\neq (n-1)\cdot M$$

$$\tilde{x}[n] = x[nM]$$

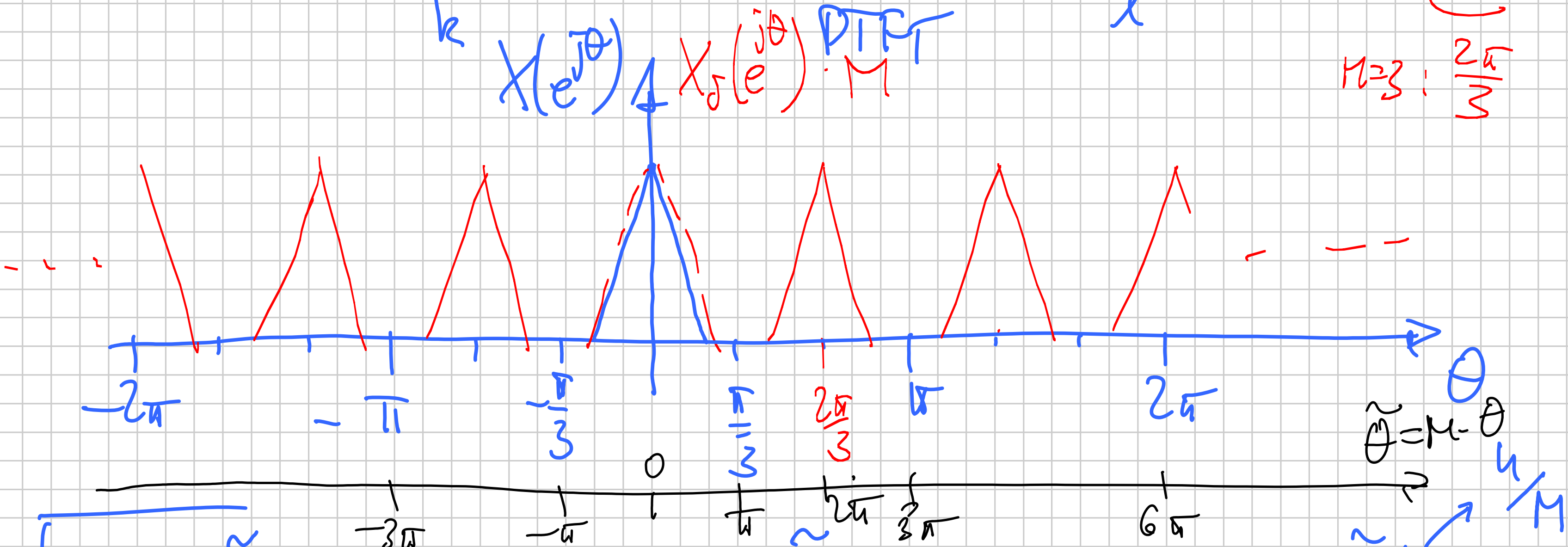


$$x_\delta[n] = \sum_k x[k]\,\delta[n-kM]$$

Übergang auf DTFT

$$M\sum_k \delta[n-kM] \circ\!\!-\!\!\bullet\ 2\pi\sum_\ell \delta\left(\theta-\ell\frac{2\pi}{M}\right)$$

$M=3:\ \frac{2\pi}{3}$



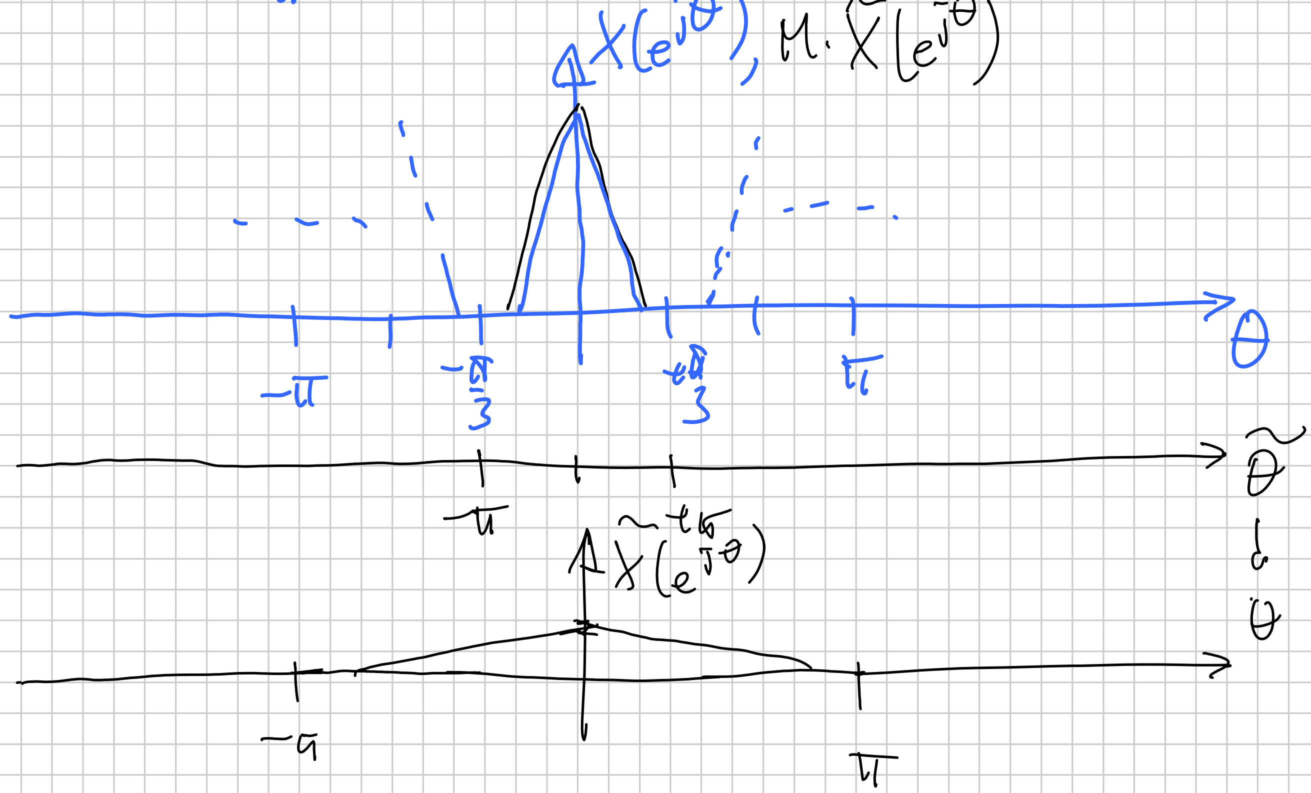
$$\boxed{\tilde X(e^{j\tilde\theta})} = \sum_k \tilde x[k]\, e^{-j\tilde\theta k} = \sum_k x_\delta[\underbrace{M\cdot k}_{n}]\, e^{-j\tilde\theta \underbrace{k}_{n/M}} =$$

$$= \sum_n x_\delta[n]\, e^{-j\tilde\theta\cdot\frac{n}{M}} =$$

$$= \sum_n x_\delta[n]\, e^{-j\frac{\tilde{\theta}}{M}\cdot n}$$

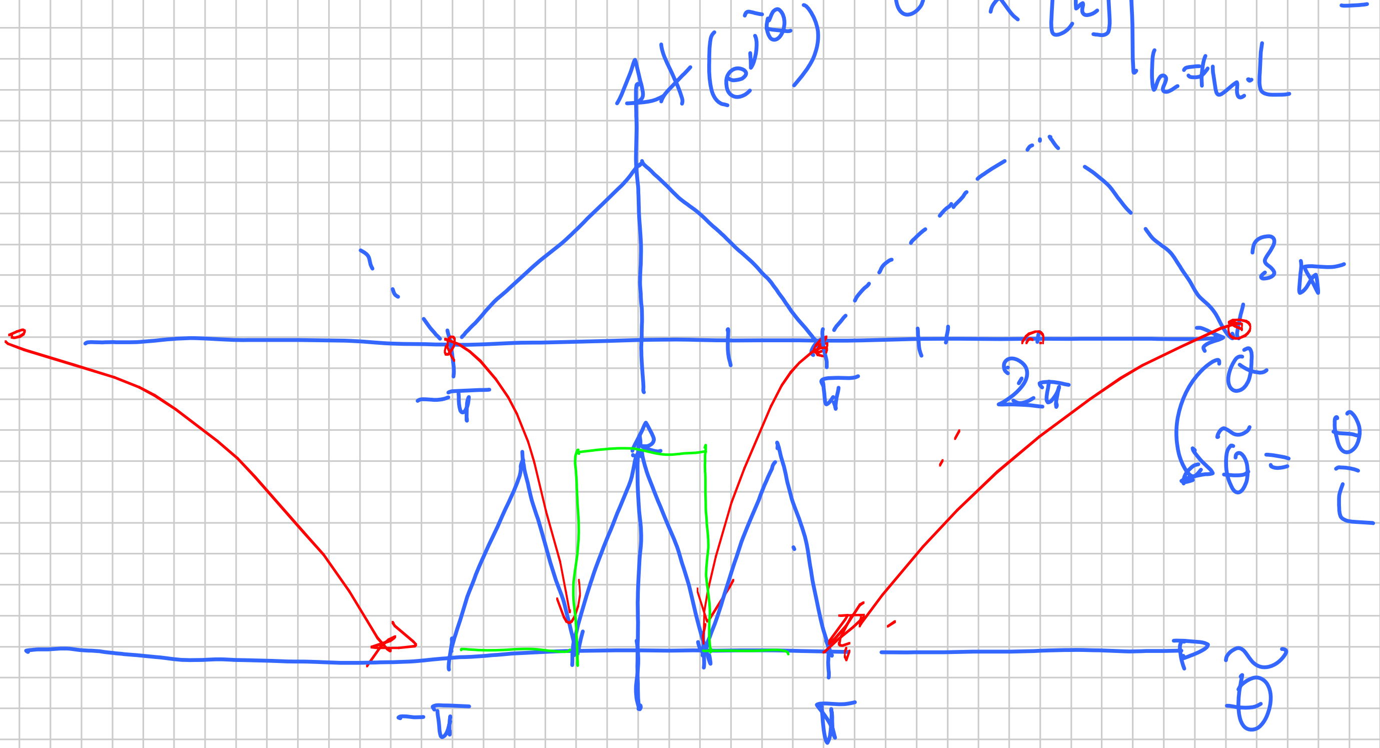
$$= \sum_n x_\delta[n]\, e^{-j\theta\cdot n} = X_\delta(e^{j\theta})$$

$$\boxed{\theta = \frac{\tilde{\theta}}{M}}$$



Taktratenerhöhung

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \tilde{x}[k] := \begin{cases} \tilde{x}[k]\big|_{k=n\cdot L} = x[n] \\ \tilde{x}[k]\big|_{k\neq n\cdot L} = 0 \end{cases}$$



$\boxed{L=3}$

$z = e^{j\theta}$ (am Einh.kreis) : $\theta \to \tilde{\theta} = \theta \cdot M$: $e^{j\theta} \to e^{j\theta M}$: $z^{1} \to z^{M}$

$\theta \to \tilde{\theta} = \theta / L$: $z^{1} \to z^{1/L}$

Interpolationsfilter:

$$x \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\text{Interpolation Tiefpass Filter}} \longrightarrow$$

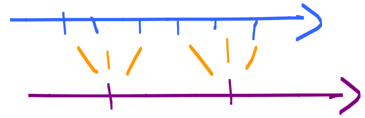
π/L Grenzfrequenz

Sampling rate conversion

- Conversion by small integers
- Rational conversion factors
- General conversion factors, fractional delays, Lagrange interpolation

$\xrightarrow{16\,kHz} \boxed{TP}^{\pi/6} \rightarrow \boxed{\downarrow M} \rightarrow$ $M=6$ $8\,kHz$

↳ z.B.: DAC von 32 kHz → ADC mit 8 kHz



Interpolation von mehreren Samples auf eines

↳ M: 16 kHz → 24 kHz: $L = \frac{3}{2}$

$\rightarrow \boxed{\uparrow 3} \rightarrow \underbrace{\boxed{\Theta_c = \frac{\pi}{3}} \rightarrow \boxed{\cancel{\Theta_c = \frac{\pi}{2}}}} \rightarrow \boxed{\downarrow 2}$

Es sind nicht beide notwendig, das stärkere belassen

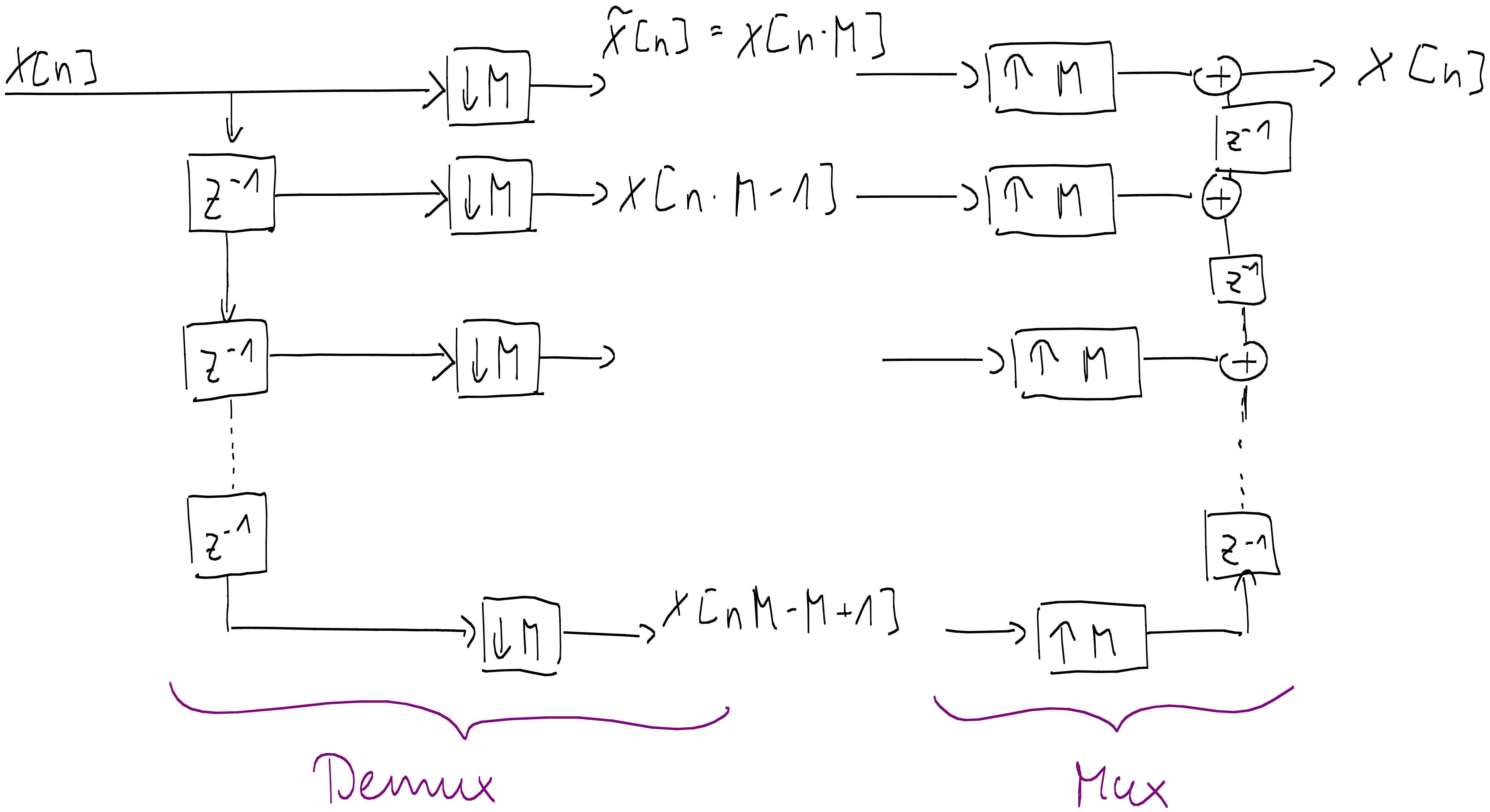
z.B.: $M = 128 \Rightarrow \Theta_c = \frac{\pi}{128}$ … sehr aufwendig da lange Impulsantwort

$\Rightarrow$ Lösung $M_1 \cdot M_2$ z.B.: $16 \cdot 8$

$x[n] \rightarrow \boxed{\Theta_c = \frac{\pi}{8}} \rightarrow \boxed{\downarrow 8} \rightarrow \boxed{\Theta_c = \frac{\pi}{16}} \rightarrow \boxed{\downarrow 16} \rightarrow \tilde{x}[n]$

Höheres Filter erst später einsetzen!

Polyphasenzerlegung:



$$(x * h)[n] = y[n]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{\downarrow M} \longrightarrow (x * h)[k \cdot M] = y[kM]$$

$$y[kM] \to \tilde{y}[k]$$

$$y[n]\Big|_{k = k \cdot M} = y[k \cdot M] = \sum_m x[m]\, h[n-m]\Big|_{n = k \cdot M}$$

$$= \sum_m x[kM - m]\, h[m]$$

$$m = p \cdot M + \ell \leftarrow \text{"Teilphase"}$$

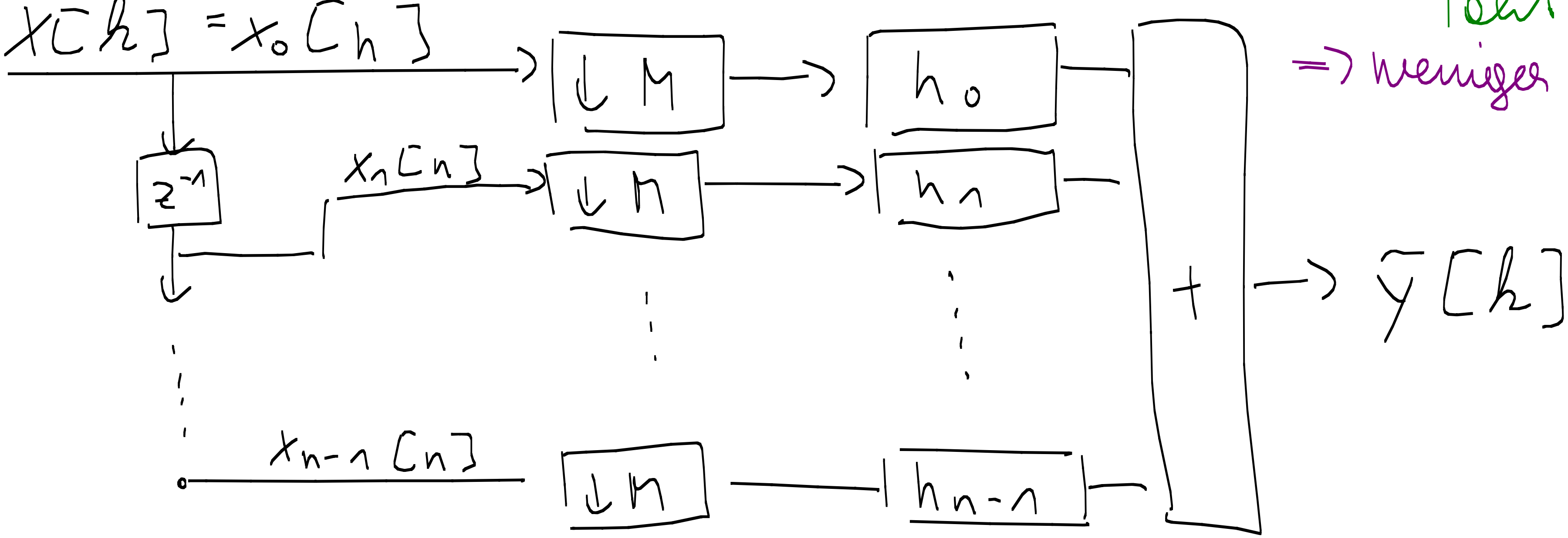
$$y[kM] = \sum_{p=-\infty}^{\infty} \sum_{\ell=0}^{M-1} x[kM - pM - \ell]\, h[pM + \ell]$$

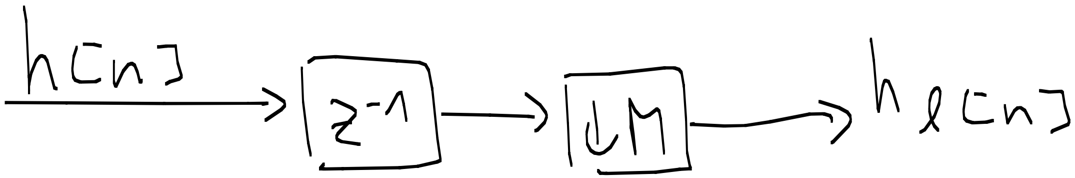
$$= \sum_{p=-\infty}^{\infty} \sum_{\ell=0}^{M-1} x_\ell[(k-p)M]\, h_\ell[pM]$$

$$\tilde{y}[k] = y[kM] = \sum_{l=0}^{M-1} (x_l * h_l)[k]$$

M Filter um eine um $\frac{1}{M}$ reduzierte Länge der Impulsantwort auf $\frac{1}{M}$ fach Takt

⇒ weniger Aufwand





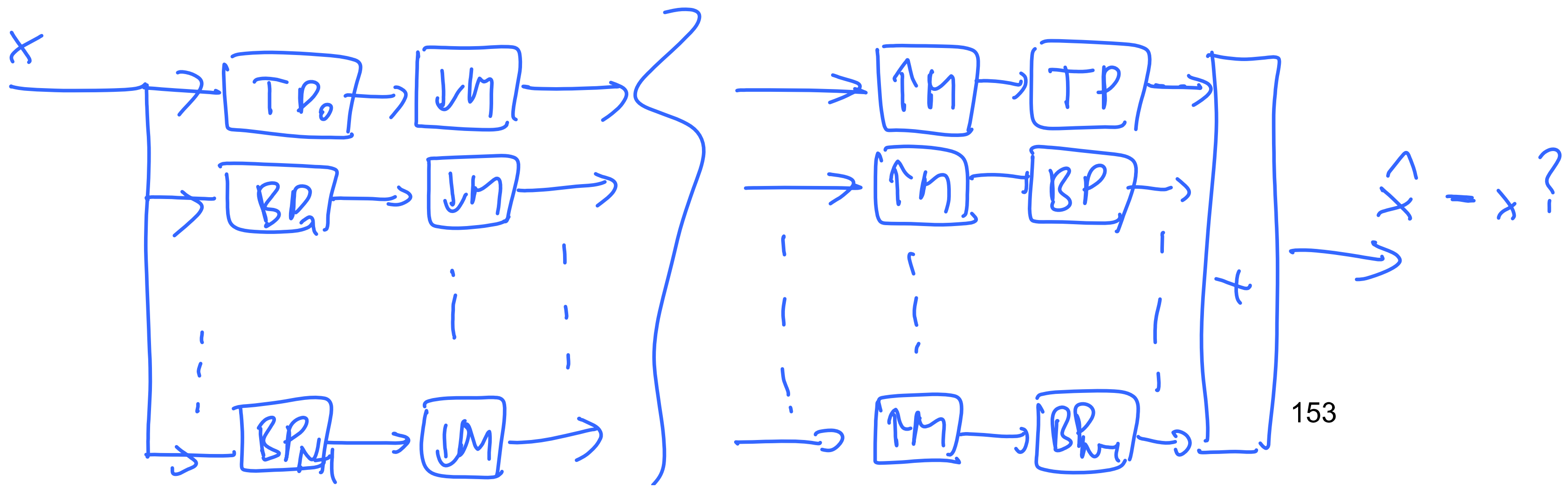
"Teilimpulsantwort"

Filterbanks and polyphases

- Analysis-synthesis filterbanks
- Subsampling of filterbank channels: critical sampling vs. oversampled filterbanks

$N > M$ $N = M$

- Polyphase representation, general case
- Detailed example: two channel case, odd and even sample decomposition



QMF and perfect reconstruction

- Subsampling with non-ideal bandpass filters results either in aliasing or in spectral gaps at band transitions
- Design band edges such that aliasing components of adjacent bands cancel
- Two-channel case: quadrature-mirror filter design

Perfect reconstruction, cont'd

- General case: all aliasing components from M channels may overlap $\rightarrow$ matrix algebraic problem where matrix elements are polynomials in z^{-1} $\rightarrow$ *perfect reconstruction* in case of unmodified back-to-back connection
- Non-unique solution for synthesis filters $\rightarrow$ minimize noise gain of synthesis filters; relax to *near perfect reconstruction*