Bestimmung der diagonalisierenden Matrix Q:

Beispiel:

$$A = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 2 & -2 \\ 4 & 2 & -1 \end{pmatrix} \quad \text{reell, symetrisch}$$

Eigenwerte: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 4 \\ 2 & 2-\lambda & -2 \\ 4 & -2 & -1-\lambda \end{vmatrix}$

$$= -\lambda^3 + 27\lambda - 54 \overset{!}{=} 0$$

$\lambda_1 = 3$ (geraten)

$$\left.\begin{array}{l} \lambda_2 = 3 \\ \lambda_3 = 6 \end{array}\right\} \text{polynomdivision}$$

EV zu $\lambda_{1,2} = 3$: $(A - 3I) \cdot v = \begin{pmatrix} -4 & 2 & 4 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rightsquigarrow \begin{pmatrix} 2 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$2x_1 - x_2 - 2x_3 = 0$$

wähle $x_2 = s$, $x_3 = t$

$$\Rightarrow \quad x_1 = \frac{s}{2} + t$$

setze $s=1,\ t=0$

$$v_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

setze $s=0,\ t=1$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

EV zu $\lambda_3 = -6$ $\quad (A-(-6)I)v \stackrel{!}{=} 0$

$$\begin{pmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 4 & -1 \\ 0 & -18 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 + 4x_2 - x_3 = 0$$
$$-18x_2 + 9x_3 = 0$$

wähle $x_3 = 2t$

$$\Rightarrow v_3 = t \cdot \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Eigenraum: $E_3 = L\left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$E_{-6} = L\left\{ \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Normalisieren:

$$w_1 = \frac{1}{\|v_1\|} \cdot v_1 = \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$u_2 = v_2 - \langle v_2, w_1 \rangle \cdot w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{\frac{1}{5} \cdot 1 \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 4/5 \\ -2/5 \\ 1 \end{pmatrix} \hat{=} \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$$

$$w_2 : \frac{1}{\|v_2\|} \cdot u_2$$

$$= \frac{1}{3\sqrt{5}} \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$$

$$w_3 : \frac{1}{\|v_3\|} \cdot v_3$$

$$= \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & -\frac{1}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3} \end{pmatrix}; \quad Q^T \cdot A \cdot Q = \ldots = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

Beispiel: Man berechne A^{10} für

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$C^{-1} \cdot A \cdot C = D \Leftrightarrow A = C \cdot D \cdot C^{-1}$$

$$A^n = C \cdot D^n \cdot C^{-1}$$

EW von A:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1) \overset{!}{=} 0$$

$$\lambda_1 = 2$$

$$\lambda_2 = -1$$

EV zu $\lambda_1 = 2$: $(A - 2I) \cdot v \overset{!}{=} 0$

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 = 0$$

wähle $x_2 = 2t \Rightarrow x_1 = t$

$$v_1 = t \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ t \in \mathbb{R}$$

EV zu $\lambda_2 = -1$ $(A - (-1)I) \cdot v \overset{!}{=} 0$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 = 0$$

$$v_2 = t\begin{pmatrix}1\\-1\end{pmatrix}$$

$$C = \begin{pmatrix}1 & 1\\2 & -1\end{pmatrix} \qquad C^{-1} = \frac{1}{\det(C)}\begin{pmatrix}-1 & -1\\-2 & 1\end{pmatrix}$$

$$\det(C) = -3 \qquad = \frac{1}{3}\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}$$

$$D = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}, \quad D^2 = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}2^2 & 0\\0 & (-1)^2\end{pmatrix}$$

$$D^3 = D\cdot D^2 = \begin{pmatrix}2^2 & 0\\0 & (-1)^2\end{pmatrix}\begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}2^3 & 0\\0 & (-1)^3\end{pmatrix}$$

$$D^{10} = \begin{pmatrix}2^{10} & 0\\0 & (-1)^{10}\end{pmatrix}$$

$$A^{10} = C\cdot D^{10}\cdot C^{-1} = \begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}2^{10} & 0\\0 & (-1)^{10}\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}$$

$$= \begin{pmatrix}342 & 341\\682 & 683\end{pmatrix}$$