# linA KV

Note Title 12.12.2007

## Beispiel 1.)

$$Ax = b$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

3 Schritte mit 1.) Jakobi
2.) Gauss-Seidel

$$A = S - T$$

$$Sx - Tx = b$$

$$Sx_{k+1} = Tx_{k+1} + b$$

$$x_{k+1} = S^{-1}Tx_k + S^{-1}b$$

### 1.) Jakobi

$$S = \operatorname{diag}(A) : \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T = S - A : \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1}T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$S^{-1}b = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_{21} \\ x_{22} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

mit 2.) Gaus Seidl

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad S^{-1}b = \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix}$$

$$T = S - A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S^{-1}T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_{21} \\ x_{22} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\begin{pmatrix} x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

Exakte Lösung da es in 2 Schritten das selbe Ergebnis gab.

## Beispiel 2.)

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 2 & 2 \end{pmatrix}$$

Pseudoinverse berechnen:

$$A^{\#} = (A^T A)^{-1} A^T$$

$$\cdot)\ A^T A = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 10 \\ 10 & 14 \end{pmatrix}$$

$$\cdot)\ (A^T A)^{-1} = \frac{1}{96} \begin{pmatrix} 14 & -10 \\ -10 & 14 \end{pmatrix}$$

$$\cdot)\ (A^T A)^{-1} A^T = \frac{1}{96} \begin{pmatrix} 14 & -10 \\ -10 & 14 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix} = \frac{1}{96} \begin{pmatrix} -16 & 32 & 8 \\ 32 & -16 & 8 \end{pmatrix} = A^{\#}$$

$$\Rightarrow \quad Ax = b$$

$$x = A^{-1} b$$

$$A^{-1} \nexists \cdot \quad \bar{x} = A^{\#} b$$

Beispiel 3.)

Es sind 2 Zeilen gleich $\Rightarrow \nexists A^{-1}$

$\nexists A^{-1}$ :

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0{,}99 \\ 1{,}01 \\ 1 \end{pmatrix}$$

$$(A^TA) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (A^TA)^{-1} \nexists$$

SVD: $A = U \Sigma V^T$

$$A^\# = V \Sigma^\# U^T \qquad \Sigma^\# = \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

EW: $$\det(A^TA - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2(1-\lambda) - 4(1-\lambda) \overset{!}{=} 0$$

$\lambda_1 = 1 \Rightarrow \sigma_2 = 1$

$\lambda_2 = 0 \Rightarrow (\sigma_3 = 0)$ $\quad \sigma \neq 0$

$\lambda_3 = 4 \qquad \sigma_1 = \sqrt{4} = 2$

$$\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

EV $(A^TA)$: $$\lambda = 4: \begin{pmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x_3 = 0$; $x_2 = x_1$; $\Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda = 1: \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{array}{l} x_1 = x_2 = 0 \\ x_3 \text{ beliebig} \end{array}$$

$$\lambda = 0: \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{array}{l} x_1 = -x_2 \\ x_3 = 0 \end{array}$$

sie sind schon orthogonal;
sonst (Gram-Schmidt)

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$U: \; u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{1}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$u_3$ durch Kreuzprodukt ermitteln
(es ist ein 3 Vektor l.u erforderlich,
das Kreuzprodukt liefert solch einen)

$$u_3 = u_1 \times u_2 = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Normieren zu

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^\# = V \Sigma^\# U^T$$

$$\Sigma^\# = \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow D^{-1} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Sigma^\# = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$