LinA · Tut

Note Title 21.11.2007

$$v_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix},\quad v_2 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix},\quad v_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix},\quad v_4 = \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

ges.: orthonormalsystem

$$w_1 = \frac{v_1}{\|v_1\|} \;;\; w_i = \frac{u_i}{\|u_i\|} \;;\; u_i = v_i - \sum_{k}^{i-1} \langle v_i, w_k \rangle w_k$$

$$\|v\| = \sqrt{\langle v, v\rangle}$$

- $w_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

- w_2: $u_2 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \underbrace{\langle v_2, w_1 \rangle}_{\frac{1}{\sqrt{2}}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

$$= \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 1\\ 0\end{pmatrix}$$

$$w_2 = \frac{u_2}{\|u_2\|} = \sqrt{\frac{2}{3}} \cdot \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 1\\ 0\end{pmatrix}$$

$$\bullet\ w_3: \quad u_3 = v_3 - \overbrace{\underbrace{\langle v_3, w_1\rangle}}^{0}\cdot w_1 - \underbrace{\langle w_3, w_2\rangle}\cdot w_2$$

$$\left\langle \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\rangle \qquad \left\langle \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\\0\end{pmatrix}\right\rangle = \sqrt{\frac{2}{3}}$$

$$= \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \sqrt{\frac{2}{3}}\cdot\sqrt{\frac{2}{3}}\cdot\begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix} = \begin{pmatrix}1/3\\-1/3\\1/3\\1\end{pmatrix}$$

$$\|u_3\| = \frac{1}{\sqrt{\frac{2}{3}}}$$

$$w_3 = \frac{u_3}{\|u_3\|} = \sqrt{\frac{3}{2}}\begin{pmatrix}\frac{1}{3}\\-\frac{1}{3}\\1/3\\1\end{pmatrix}$$

$$\bullet\ w_4: \quad u_4 = v_4 - \underbrace{\langle v_4, w_1\rangle}_{\mathrm{I}}\cdot w_1 - \underbrace{\langle v_4, w_2\rangle}_{\mathrm{II}}\cdot w_2 - \underbrace{\langle v_4, w_3\rangle}_{\mathrm{III}}\cdot w_3$$

$$\mathrm{I.)}\ \left\langle \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\rangle = \frac{1}{\sqrt{2}}$$

$$\mathrm{II}\ \left\langle \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix}\right\rangle = -\frac{1}{2}\sqrt{\frac{2}{3}}$$

$$\mathrm{III}\ \left\langle \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \sqrt{\frac{3}{2}}\begin{pmatrix}1/3\\-1/3\\1/3\\1\end{pmatrix}\right\rangle = \sqrt{\frac{3}{2}}\left(-\sqrt{\frac{2}{3}}\right) = \frac{\sqrt{3}}{3}$$

$$u_4 = \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\\0\end{pmatrix} + \frac{\cancel{\sqrt{3}}}{\cancel{3}}\cdot\frac{\cancel{\sqrt{3}}}{2}\begin{pmatrix}1/3\\-1/3\\1/3\\1\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\1/2\\-1/2\end{pmatrix}$$

$$w_4 = \frac{u_4}{\|u_4\|} = \begin{pmatrix}1/2\\-1/2\\1/2\\-1/2\end{pmatrix}$$

Beispiel 2:

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t) \, dt$$

$\mathbb{P}_2$: ges.: Orthonormierung $\begin{Bmatrix} p_1, & p_2, & p_3 \\ 1, & t, & t^2 \end{Bmatrix}$

- w_1: $w_1 = \frac{p_1}{\|p_1\|} = 1$

- w_2: $u_2 = p_2 - \langle p_2, w_1 \rangle \cdot w_1$

$$\langle p_2, w_1 \rangle = \int_0^1 t \, dt = \frac{1}{2}$$

$$u_2 = t - \frac{1}{2}$$

$$w_2 = \frac{u_2}{\|u_2\|} : \langle t - \frac{1}{2}, t - \frac{1}{2} \rangle = \int_0^1 (t^2 - t + \frac{1}{4}) \, dt$$

$$= \frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{4} t \Big|_0^1$$

$$= \frac{1}{12}$$

w_3: $\frac{u_3}{\|u_3\|}$ $\quad \langle u_3, u_3 \rangle = \int_0^1 (t^2 - t + \frac{1}{6})^2 \, dt = \frac{1}{180}$

$$w_3 = \sqrt{180} \cdot (t^2 - t + \frac{1}{6})$$

Beispiel 3.)

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

ges.: QR - Zerlegung

Q... orthonormierte Spalten
R... obere Dreiecksmatrix

$$a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$A \cdot x = b$$
$$QRx = b \qquad Q^{-1} = Q^T$$
$$Rx = Q^T \cdot b$$

$$a_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

w_1: $$w_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

w_2: $$u_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - \frac{4}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ -\frac{1}{\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$A = Q \cdot R \Rightarrow R = Q^T \cdot A$$

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ 0 & 3 \end{pmatrix}$$