Beispiel 1:

$F: \mathbb{P}_2 \to \mathbb{P}_2$

$p(x) \mapsto F(p(t)) := p(2t-1)$

ges.: $\cdot\ M_K^K$ $\qquad K = (1, t, t^2)$ $\quad$ $k_1\ k_2\ k_3$

$\cdot$ Ker(F)

$\cdot$ Bild(F)

$\cdot$ Injektiv / surjektiv / bijektiv ?

$F(1) = F(t^0) = (2t-1)^0 = 1 \qquad = 1 \cdot k_1$

$F(t) = F(t^1) = (2t-1)^1 = 2t-1 \qquad = -1\, k_1 + 2\, k_2$

$F(t^2) = F(t^2) = (2t-1)^2 = 4t^2 - 4t + 1 \qquad = 1\, k_1 - 4\, k_2 + 4\, k_3$

$$M_K^K(F) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{pmatrix}$$

Kern(F): $\quad p(t): F(p(t)) = 0$

$p(t) = a + bt + ct^2$ ... allgemeines $\mathbb{P}_2$

$F(p(t)) = a \cdot 1 + b(2t-1) + c(2t-1)^2$

$= (a - b + c) + t(2b - 4c) + t^2(4c) \overset{!}{=} 0$

$$\left.\begin{aligned} &\Rightarrow a - b + c \overset{!}{=} 0 \\ &\Rightarrow 2b - 4c \overset{!}{=} 0 \\ &\Rightarrow 4c \overset{!}{=} 0 \end{aligned}\right\} a = 0;\ b = 0;\ c = 0$$

$$\Rightarrow \text{Kern}(F) = 0 \quad \forall\ a, b, c = 0$$

- $\text{Kern}(F) = 0 \Leftrightarrow$ injektiv

$\dim(\mathbb{P}_2) = 3;\quad \dim(\mathbb{P}_3) = 4;\quad \dim(\mathbb{R}^2) = 2,\quad \dim(\mathbb{R}^3) = 3$

Dimensionsformel: $\dim(\text{Kern}) + \dim(\text{Bild}) = \dim(W)$

$$\left.\begin{aligned} \dim(\text{Kern}(F)) &= 0 \\ \dim(W) &= 3 \end{aligned}\right\} \dim(\text{Bild}(F)) = 3$$

- $\dim(\text{Bild}(F)) = \dim(W) \Rightarrow$ surjektiv

  injektiv & surjektiv = bijektiv

Bild(F):

Bilder in Zeilen:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -4 & 4 \end{pmatrix} \begin{matrix} (+1) \\ (-1) \end{matrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -4 & 4 \end{pmatrix} \; (+2)$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\ \begin{array}{ccc} t^0 & t^1 & t^2 \end{array} \end{array} \Rightarrow \text{Bild}(F) = \{1, 2t, 4t^2\}$$

$=$

$$K' = \{\underbrace{1+t}_{K_1'}, \underbrace{1-t}_{K_2'}, \underbrace{1+t+t^2}_{K_3'}\}$$

gleiches Bsp, andere Basis

ges.: $M^{K'}_{K'}(F)$

$$F(1+t) = 1+2t-1 = 2t = \underbrace{(1+t)}_{K_1'} + \underbrace{(1-t)}_{K_2'} = 2t$$

$$F(1-t) = 1-2t+1 = 2-2t = 2\,K_2'$$

$$F(1+t+t^2) = 1+2t-1+4t^2-4t+1 = 4t^2-2t+1$$

GLS um Koeffizienten zu ermitteln $\Leftarrow \lambda_1 K_1' + \lambda_2 K_2' + \lambda_3 K_3'$

$$= \lambda_1(1+t) + \lambda_2(1-t) + \lambda_3(1+t+t^2)$$

$$4t^2 - 2t + 1 = \lambda_3 t^2 + t(\lambda_1 - \lambda_2 + \lambda_3) + 1(\lambda_1 + \lambda_2 + \lambda_3)$$

$$\Rightarrow \lambda_3 = 4, \quad \lambda_2 = 3/2, \quad \lambda_1 = -9/2$$

$$M^{K'}_{K'}(F) = \begin{array}{c} \begin{pmatrix} 1 & 0 & -9/2 \\ 1 & 2 & 3/2 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{c} K_1' \\ K_2' \\ K_3' \end{array} \\ \begin{array}{ccc} F(1+t) & F(1-t) & F(1+t+t^2) \end{array} \end{array}$$

Beispiel 2:

$F: \mathbb{P}_2 \to \mathbb{R}^3$

$$F(a + bt + ct^2) = \begin{pmatrix} a+b \\ b+c \\ c-a \end{pmatrix}$$

ges.) $M^A_B(F)$; $A = \{1, t, t^2\}$

$B = \{e_1, e_2, e_3\}$

$$F(1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = e_1 - e_3$$

$$F(t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = e_1 + e_2 \qquad \Rightarrow M^A_B(F) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$F(t^2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = e_2 + e_3$$

ges $p(t) = 2 + t - t^2$

$w = F(p(t))$

1.) direkt $F(2 + t - t^2) = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

2.) $c_B(w) = A \cdot c_A(p)$

$$c_A(p) = 2 + t - t^2 = \lambda_1 \cdot 1 + \lambda_2 \cdot t + \lambda_3 t^2$$

Basis

c_A

$$c_A(p) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$ aus $p(t)$ leicht erkennbar da kanonische Basis

$$c_B(p) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$$

$$\Rightarrow v = \langle c_B(v), B \rangle = 3e_1 + 0e_2 - 3e_3 = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$$

Prüfungshinweis:

- GLS mit Unbekannten Lösung; (gibt es eine Lösung; eindeutig lösbar)
- LR Zerlegung; als GLS
- Det von Matrix, Matrix^{-1}; für welche $a \in \mathbb{R}$ ist die Matrix invertierbar
- $M_B^A(F)$;
- Bild (F); Kern (F)
- c_A, c_B;