lineare Algebra Hausübung 02:

Altinger Harald
0630936
Gruppe 01

Note Title 06.01.2008

1a) Ermitteln der Eigenwerte:

$$A_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P(\lambda) = \det(A_1 - \lambda \cdot I)$$

$$\left| \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right|$$

$$\left| \begin{pmatrix} -(1+\lambda) & 0 & 1 \\ 0 & -(1+\lambda) & 0 \\ 1 & 0 & -(1+\lambda) \end{pmatrix} \right| \begin{matrix} -(1+\lambda) & 1 \\ 0 & -(1+\lambda) \\ 1 & 0 \end{matrix}$$

$$\det = 0 + 0 + 0 - (-(1+\lambda))^3 - 0 - 0 \overset{!}{=} 0$$

$$= (1+\lambda)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$$

$$\Rightarrow \lambda_i = -1$$

Ermittlung der Eigenvektoren zu $\lambda = -1$

$$(A_1 - \lambda I)v = 0$$

$$(A_1 - (-1)I) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\operatorname{rang}(A_1 - (-1)I) = 2 \neq 3$$

$$\left.\begin{aligned} x_3 + 0 + 0 &= 0 \\ 0 + 0 + x_2 &= 0 \\ 0 + 0 + 0 &= 0 \end{aligned}\right\} \Rightarrow v_1 = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$x_1 = t$$

$$v_2 = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

1b.) Q ... orthonormiert

laut Gram-Schmidt:

$$w_1 = \frac{1}{\|v_1\|} \cdot v_1 \qquad v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\|v_1\| = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{\sum_{i=1}^{3} x_i \cdot x_i} = \sqrt{1} = 1$$

$$w_1 = \frac{1}{1} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$w_2 = \frac{1}{\|u_2\|} \cdot u_2$$

$$u_2 = v_2 - \sum_{k=1}^{2-1} \langle v_2, w_k \rangle \cdot w_k$$

$$= \begin{pmatrix}0\\1\\0\end{pmatrix} - \underbrace{\left\langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle}_{0} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\|u_2\| = \sqrt{0 + 1 + 0} = 1$$

$$w_2 = \frac{1}{1} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$v_3 = \frac{1}{\|u_3\|} \cdot u_3$$

$$u_3 = v_3 - \sum_{k=1}^{2} \langle v_3, v_k \rangle v_k$$

$$= v_3 - \langle v_3, v_1 \rangle v_1 - \langle v_3, v_2 \rangle v_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle}_{0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle}_{0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\|u_3\| = \sqrt{0 + 0 + 1} = 1$$

$$v_3 = \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q^{-1} = Q^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \mathrm{diag}(\lambda) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$Q^T \cdot A_1 \cdot Q = D$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2.) a.) QR- Zerlegung:

$$M_1 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \\ 2 & 0 \end{pmatrix}$$

$$m_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \qquad m_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$(-1)\ \begin{pmatrix} 2 & 1 \\ 1 & -2 \\ 2 & 0 \end{pmatrix} \left(-\tfrac{1}{2}\right) \rightsquigarrow \begin{pmatrix} 2 & 1 \\ 0 & -5/2 \\ 0 & -1 \end{pmatrix}$$

rang = 2 $\Rightarrow m_1, m_2$ l.u.

orthonormieren nun m_1, m_2:

$$n_1 = \frac{1}{\|m_1\|} \cdot m_1$$

$$\|m_1\| = \sqrt{\langle m_1, m_1 \rangle} = \sqrt{4+1+4} = 3$$

$$= \frac{1}{3} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$w_2 = \frac{1}{\|u_2\|} \cdot u_2$$

$$u_2 = m_2 - \langle m_2, w_1 \rangle w_1$$

$$= \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle}_{1 \cdot \frac{2}{3} + (-2) \frac{1}{3} + 0 \; = \; 0} \cdot \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\|u_2\| = \sqrt{\langle u_2, u_2 \rangle} = \sqrt{1+4} = \sqrt{5}$$

$$w_2 = \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 2/3 & 1/\sqrt{5} \\ 1/3 & -2/\sqrt{5} \\ 2/3 & 0 \end{pmatrix} \Rightarrow Q^T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \end{pmatrix}$$

$$
\begin{array}{c|cc}
 & 2 & 1 \\
 & 1 & -2 \\
 & 2 & 0 \\
\hline
R = Q^T \cdot M_1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \end{pmatrix} & \frac{4}{3}+\frac{1}{3}+\frac{4}{3} & \frac{2}{3}-\frac{2}{3}+0 \\
 & \frac{2}{\sqrt{5}}-\frac{2}{\sqrt{5}}+0 & \frac{1}{\sqrt{5}}+\frac{4}{\sqrt{5}}+0
\end{array}
$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 5/\sqrt{5} \end{pmatrix}$$

2b.) Psydoinverse:

$$M^{\#} = (M^{+} M)^{-1} \cdot M^{+}$$

$$M^T = \begin{pmatrix} 2 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix}$$

$$
\begin{array}{ccc|cc}
 & & & 2 & 1 \\
 & & & 1 & -2 \\
 & & & 2 & 0 \\
\hline
M^T \cdot M = \; 2 & 1 & 2 & 4+1+4 & 2-2\;0 \\
1 & -2 & 0 & 2-2+0 & 1+4+0
\end{array}
$$

$$= \begin{matrix} 9 & 0 \\ 0 & 5 \end{matrix}$$

$$(M^T \cdot M)^{-1}: \left(\begin{array}{cc|cc} 9 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/9 & 0 \\ 0 & 1 & 0 & 1/5 \end{array}\right)$$

$$\begin{array}{c|c} & \begin{pmatrix} 2 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix} \\ \hline M^{\#} = \begin{pmatrix} 1/9 & 0 \\ 0 & 1/5 \end{pmatrix} & \begin{matrix} 2/9+0 & 1/9 & 2/9 \\ 0+1/5 & 0-2/5 & 0 \end{matrix} \end{array}$$

$$M^{\#} = \begin{pmatrix} 2/9 & 1/9 & 2/9 \\ 1/5 & -2/5 & 0 \end{pmatrix}$$