Differentialgleichungen Ue

Note Title 28.04.2008

Lineare Dgl.

15a.) $x'' + 4x' + 3x = 0$ $x(t) = ?$

$x = c e^{\lambda t}$

charakteristisches Polynom:

$$\lambda^2 + 4\lambda^1 + 3\lambda^0 = 0$$

$$\lambda_{1,2} = -2 \pm \sqrt{4-3} = \begin{cases} -1 \\ -3 \end{cases}$$

$$\Rightarrow x(t) = c_1 e^{-1t} + c_2 e^{-3t}$$

15b.) $x''' - x'' + x' - x = 0$

$\lambda^3 - \lambda^2 + \lambda - 1 = 0 \Rightarrow +1$ ist Nullstelle

Teiler als Nullstelle

$$\begin{array}{l} (\lambda^3 - \lambda^2 + \lambda - 1)(\lambda - 1) = \lambda^2 + 1 \\ \underline{-\lambda^3 + \lambda^2} \\ \quad 0 \; R \end{array}$$

$\lambda^2 = -1 \Rightarrow \lambda = \pm j$

Eigenwerte des char. Polynoms: $\lambda_1 = 1$

$\lambda_2 = i$

$\lambda_3 = -i$

$$e^{it} = \cos(t) + i\sin(t)$$
$$e^{-it} = \cos(t) - i\sin(t)$$

Lösung: $x(t) = c_1 e^t + c_2\cos(t) + c_3\sin(t)$

AWP:

allgemein: $\lambda = \alpha + i\beta$

$$\Rightarrow \underbrace{c_1 e^{\alpha t}\cos(\beta t)}_{\text{alle reellen}} + \underbrace{c_2 e^{\alpha t}\sin(\beta t)}_{\text{alle imaginären}}$$

$$x(0) = c_1 \cdot 1 + c_2 \cdot 1 + c_3 \cdot 0 \overset{!}{=} 1$$

$$\dot{x}(t) = c_1 e^t - c_2\sin(t) + c_3\cos(t)$$

$$\dot{x}(0) = c_1 - 0 + c_3 \overset{!}{=} 0$$

$$\ddot{x}(t) = c_1 e^t - c_2\cos(t) - c_3\sin(t)$$

$$\ddot{x}(t) = c_1 - c_2 - 0 \overset{!}{=} -1$$

$$2c_1 = 0 \Rightarrow c_1 = 0$$

$$c_2 = 1$$

$$c_3 = 0$$

$$\Rightarrow x(t) = 1 \cdot \cos(t)$$

Funktioniert nur wenn $c_i \neq f(t)$!

15c.) $y^{IV} + y = 0$

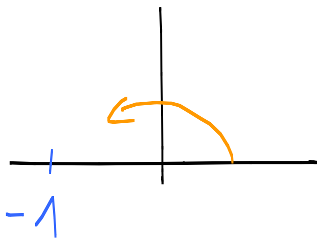
$$\lambda^4 + 1 = 0$$

$$\mu = \lambda^4 = -1$$

in Polarcoord

$\hookrightarrow |\mu| = 1$

$$\varphi = \arg(\mu) = \pi$$



$$\sqrt[4]{\mu} = \sqrt[4]{|\mu|}\left(\cos\frac{\varphi + 2k\pi}{4} + i \sin\frac{\varphi + 2k\pi}{4}\right) \quad k = 0 \ldots 3$$

$$k = 0: \quad \lambda_1 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}$$

$$k = 1: \quad \lambda_2 = +\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}$$

$$\lambda_2 = \overline{\lambda_1}$$

$$k = 2: \quad \lambda_3 = -\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}$$

$$k = 3: \quad \lambda_4 = -\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}$$

$$\lambda_3 = \overline{\lambda_2}$$

$$x(t) = c_1 e^{\frac{1}{\sqrt{2}}t} \cos\left(\frac{1}{\sqrt{2}}t\right) + c_2 e^{\frac{1}{\sqrt{2}}t} \sin\left(\frac{1}{\sqrt{2}}t\right)$$

$$+ c_3 e^{-\frac{1}{\sqrt{3}}t} \cos\left(\frac{1}{\sqrt{2}}t\right) + c_4 e^{-\frac{1}{\sqrt{2}}t} \sin\left(\frac{1}{\sqrt{2}}t\right)$$

15d.) inhomogen: $x'' + x = \sin(2t)$

Homogen $x'' + x = 0$

$$\lambda^2 + 1 = 0$$

$$x_h = c_1 \cos(t) + c_2 \sin(t)$$

Variation der Konstanten:

$$x_p = c_1(x) \cos(t) + c_2(x) \sin(t)$$

$$c_1 = \int \frac{f(t)\, x_2}{W(t)}\, dt$$

$$c_2 = \int \frac{f(t)\, x_1}{W(t)}\, dt$$

$$W(t) = \det \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1$$

$$c_1 = \int \frac{-\sin(2t)\sin(t)}{1}\, dt$$

$$= -\int 2 \sin^2(t) \cos(t)\, dt$$

$$\sin(t) = u \; ; \; \cos(t)\, dt = du$$

$$= -\int 2u^2\, du$$

$$= -\frac{2}{3} u^3 \Rightarrow -\frac{2}{3} \sin^3(t)$$

$$c_2 = \int \frac{\sin(2t)\cos(t)}{1}\,dt$$

C212

$$= -\int -2\sin(t)\cos^2(t)\,dt$$

$$\cos(t) = v$$
$$-\sin(t)\,dt = dv$$

$$= -\int 2v^2\,dv$$

$$= -\frac{2}{3}v^3 = -\frac{2}{3}\cos^3(t)$$

$$x_p = -\frac{2}{3}\sin^3(t)\cdot\cos(t) - \frac{2}{3}\cos^3(t)\sin(t)$$

$$x(t) = c_1\cos(t) + c_2\sin(t) - \frac{2}{3}\sin^3(t) - \frac{2}{3}\cos^3(t)\sin(t)$$