Dgl Rep

$$\dot{x} = A \cdot x + f(t) \qquad A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix} \qquad f(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|A - \lambda I| = (6-\lambda)(3-\lambda) - 10 = \lambda^2 - 9\lambda + 8 \stackrel{!}{=} 0$$

$$\lambda_1 = 8$$
$$\lambda_2 = 1$$

$$\lambda_1 = 8: \quad \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_1 = u_2 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1: \quad \begin{pmatrix} 5 & 2 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 5u_1 + 2u_2 = 0$$

$$v_2 = \begin{pmatrix} 1 \\ -5/2 \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} e^{8t} & e^{t} \\ e^{8t} & -5/2\, e^{t} \end{pmatrix}$$

$$\phi_h = \Phi(t)\, C$$

$$\phi_p = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, f(s)\, ds$$

$$\Phi^{-1}(t) = ?$$

$$|\Phi(t)| = -\frac{5}{2}e^{9t} - e^{9t} = -\frac{7}{2}e^{9t}$$

$$\Phi(t)^T = \begin{pmatrix} e^{8t} & e^{8t} \\ e^{t} & -5/2\, e^{t} \end{pmatrix}$$

$$\Phi(t)^{-1} = -\frac{7}{2}e^{9t}\begin{pmatrix} e^{8t} & e^{8t} \\ e^{t} & -5/2\, e^{t} \end{pmatrix} = \frac{7}{2}\begin{pmatrix} 5/2\, e^{-8t} & e^{-8t} \\ e^{-t} & e^{-t} \end{pmatrix}$$

$$\Phi(s)^{-1} \cdot f(s) = \frac{7}{2}\begin{pmatrix} 5/2\, e^{-8s} & e^{-8s} \\ e^{-s} & e^{-s} \end{pmatrix}\begin{pmatrix} e^{2s} \\ e^{2s} \end{pmatrix}$$

$$= \begin{pmatrix} 5/2\, e^{-6s} + e^{-6s} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-6s} \\ 0 \end{pmatrix}$$

$$\int_{t_0}^{t} \begin{pmatrix} e^{-6s} \\ 0 \end{pmatrix} ds = -\frac{1}{6}\begin{pmatrix} e^{-6s} \\ 0 \end{pmatrix}\Bigg|_0^t$$

$$= -\frac{1}{6}\begin{pmatrix} e^{-6t} - 1 \\ 0 \end{pmatrix}$$

$$\phi_p(t) = -\frac{1}{6}\begin{pmatrix} e^{8t} & e^{t} \\ e^{8t} & -5/2\, e^{t} \end{pmatrix}\begin{pmatrix} e^{-6t} - 1 \\ 0 \end{pmatrix}$$

$$= -\frac{1}{6}\begin{pmatrix} e^{2t} - e^{8t} \\ e^{2t} - e^{8t} \end{pmatrix}$$

$$= \frac{1}{6}\left(-e^{2t} + e^{8t}\right)\begin{pmatrix}1\\1\end{pmatrix}$$

allgemeine Lösung:

$$\phi(t) = \begin{pmatrix} e^{8t} & e^{t} \\ e^{8t} & -\frac{5}{2}e^{t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} + \frac{1}{6}\left(-e^{2t} + e^{8t}\right)\begin{pmatrix}1\\1\end{pmatrix}$$

$$\phi(0) = \begin{pmatrix} 1 & 1 \\ 1 & -5/2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - 5/2\, c_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= 0 \\ c_1 &= 1 \end{aligned}$$

Beispiel:

$$\dot{x} = A \cdot x \qquad A = \begin{pmatrix} 4 & 5 \\ -2 & -2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 5 \\ -2 & -2-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2) + 10$$

$$= \lambda^2 - 2\lambda + 2 \overset{!}{=} 0$$

$\Rightarrow \lambda_{1,2} = 1 \pm i$ konjugiert komplexe Eigenwerte $\Rightarrow$ konjugiert komplexe Eigenvektoren

$$\lambda_1 = 1+i \qquad (A - \lambda_1 I) v_1 = 0$$

$$\begin{pmatrix} 3-i & 5 \\ -2 & -3-i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2u_1 + (3+i) u_2 = 0$$

$$\left.\begin{array}{l} v_1 = \begin{pmatrix} 3+i \\ -2 \end{pmatrix} \\[3em] \lambda_2 = 1-i \qquad \Rightarrow v_2 = \begin{pmatrix} 3-i \\ -2 \end{pmatrix} \end{array}\right\} \qquad \begin{array}{l} \phi_1(t) = e^{(1+i)t} \begin{pmatrix} 3+i \\ -2 \end{pmatrix} \\[3em] \phi_2(t) = e^{(1-i)} \begin{pmatrix} 3-i \\ -2 \end{pmatrix} \end{array}$$

$$\Phi(t) = \left(\phi_1(t) \mid \phi_2(t) \right)$$

$$\phi_1(t) = e^t \left[\cos(t) + i \sin(t) \begin{pmatrix} 3+i \\ -2 \end{pmatrix} \right]$$

$$= e^t \left[\begin{pmatrix} 3\cos(t) - \sin(t) \\ -2\cos(t) \end{pmatrix} + i \begin{pmatrix} \cos(t) + 3\sin(t) \\ -2\sin(t) \end{pmatrix} \right]$$

$$\phi_2(t) = e^t \left[\begin{pmatrix} 3\cos(t) - \sin(t) \\ -2\cos(t) \end{pmatrix} - i \begin{pmatrix} \cos(t) + 3\sin(t) \\ -2\sin(t) \end{pmatrix} \right]$$

$$\left.\begin{array}{l} \phi_1(t) = \operatorname{Re}\{\phi_1(t)\} + i \operatorname{Im}\{\phi_1(t)\} \\ \phi_2(t) = \operatorname{Re}\{\phi_1(t)\} - i \operatorname{Im}\{\phi_1(t)\} \end{array}\right\}$$

sind konjugiert komplex wie erwartet

$$\phi(t) = \Phi(t)\, c$$

$$= \left(\mathrm{Re}\{\phi_1(t)\} + i\, \mathrm{Im}\{\phi_1(t)\} \;\middle|\; \mathrm{Re}\{\phi_1(t)\} - i\, \mathrm{Im}\{\phi_1(t)\} \right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \mathrm{Re}\{\phi_1(t)\} c_1 + i\, \mathrm{Im}\{\phi_1(t)\} c_1 + \mathrm{Re}\{\phi_1(t)\} c_2 - i\, \mathrm{Im}\{\phi_1(t)\} c_2$$

$$= \mathrm{Re}\{\phi_1(t)\} \underbrace{(c_1 + c_2)}_{c_1^*} + \underbrace{i(c_1 - c_2)}_{c_2^*}\, \mathrm{Im}\{\phi_1(t)\}$$

$$= \left(\mathrm{Re}\{\phi_1\}, \mathrm{Im}\{\phi_1\} \right) \begin{pmatrix} c_1^* \\ c_2^* \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} 3\cos(t) - \sin(t) , & \cos(t) + 3\sin(t) \\ -2\cos(t) , & -2\sin(t) \end{pmatrix} e^t$$

Beispiel:

$$\dot{x} = Ax \qquad A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & -\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= -\lambda(2-\lambda)(3-\lambda) = 0 \qquad \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3 \end{aligned}$$

$$\lambda_1 = 0 : (A - \lambda_1 I) v_1 = 0$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow u_1 = u_3 = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2 : (A - \lambda_2 I) v_2 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow u_3 = 0 \quad u_1 = 2u_2$$

$$\Rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 3 : \begin{pmatrix} -1 & 0 & 0 \\ 1 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow u_1 = 0 \quad 3u_2 = 2u_3$$

$$\Rightarrow v_3 = \begin{pmatrix} 0 \\ 1 \\ 3/2 \end{pmatrix}$$

$$\Rightarrow \Phi(t) = \begin{pmatrix} 0 & 2e^{2t} & 0 \\ 1 & e^{2t} & e^{3t} \\ 0 & 0 & 3/2\, e^{3t} \end{pmatrix}$$