Dgl - Rep

20.05.2025

Beispiel:

$x'' + \alpha x' + x = t^2 + \cosh(t)$; es ist α so zu bestimmen, dass Resonanz vorliegt

H.) $p(\lambda) = \lambda^2 + \alpha\lambda + x = 0 \Rightarrow \lambda_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\underbrace{\frac{\alpha^2}{4} - 1}_{\overset{!}{=}0}}$

$x_h = c_1 e^{-t} + c_2 t e^{-t}$

$\Rightarrow \alpha = \pm 2$

$\alpha = +2$ gewählt $\lambda_1 = \lambda_2 = 1$

I.) $f_1(t) = t^2$ Ansatz: $x_p = a_0 + a_1 t + a_2 t^2$

$x_p' = a_1 + 2a_2 t$

$x_p'' = 2a_2$

$\Rightarrow \quad 2a_2 + 2(a_1 + 2a_2 t) + a_0 + a_1 t + a_2 t^2 = t^2$

Koeffizienten:

$t^2 : \quad a_2 = 1$

$t^1 : \quad 4a_2 + a_1 = 0 \Rightarrow a_1 = -4a_2 = -4$

$t^0 : \quad 2a_2 + 2a_1 + a_0 = 0 \Rightarrow a_0 = 6$

$x_{p1} = t^2 - 4t + 6$

I.) $f_2(t) = \cosh(t) = \frac{1}{2}(e^t + e^{-t})$

$\bar{f}_2(t) = \frac{1}{2} e^t \Rightarrow p(1) = 1 + 2 + 1 = 4 \quad x_{p2} = \frac{1}{8} e^t$

$\bar{f}_3(t) = \frac{1}{2} e^{-t} \Rightarrow p(-1) = 0$ äußere Resonanz

$$p(1)' = 0$$
$$p(1)'' = 2 \Rightarrow x_{p3} = \frac{1}{4} t^2 e^{-t}$$

$$x_{allgem} = C_1 e^{-t} + C_2 t e^{-t} + t^2 - 4t + 6 + \frac{1}{8} e^{t} + \frac{1}{4} e t^2 e^{-t}$$

Beispiel 2.)

$$y'' - 4y' + 4y = \frac{e^{2x}}{1+x}$$

H.) $y'' - 4y' + 4y = 0$

$$p(\lambda) = \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = 2 \pm \sqrt{4-4} \Rightarrow \lambda_1 = \lambda_2 = 2$$

$$y_h = C_1 \underbrace{e^{2x}}_{y_1(x)} + C_2 \underbrace{x e^{2x}}_{y_2(x)}$$

3.) $$C_1' = \frac{-y_2(x) f(x)}{W(y_1, y_2)(x)} \qquad C_2' = \frac{y_1(x) f(x)}{W(y_1, y_2)(x)}$$

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{vmatrix}$$

$$= e^{4x}(1+2x) - e^{4x} 2x$$

$$= e^{4x}$$

$$C_1' = \frac{-x e^{2x} e^{2x}}{(1+x) e^{4x}} = \frac{-x-1}{x+1} + \frac{1}{x+1}$$

$$\Rightarrow C_1 = \int \left(1 + \frac{1}{1+x}\right) dx = x + \ln|x+1|$$

$$c_2' = \frac{e^{2x} e^{2x}}{(1+x) e^{4x}} = \frac{1}{1+x} \Rightarrow c_2 = \int \frac{1}{1+x} dx = \ln|1+x|$$

$$\Rightarrow y_p = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$

$$= (x + \ln|1+x|) e^{2x} + \ln|1+x| x e^{2x}$$

$$= e^{2x} [\ln|1+x|(1+x) + x]$$

$$y_{allgemein} = c_1 e^{2x} + c_2 x e^{2x} + e^{2x} [\ln|1+x|(1+x) + x]$$

Beispiel 3.)

$$x^2 y'' + 2xy' - 12 = \sqrt{x}$$

Euler Dgl

subst: $x = e^t$

$t = \ln(x)$

$\frac{dt}{dx} = \frac{1}{x} = e^{-t}$

$$y(x) = y(e^t) = z(t)$$

$$y'(x) = \frac{dy}{dx} = \frac{dz}{dt} \frac{dt}{dx} = \dot{z} \underbrace{\frac{1}{x}}_{e^{-t}} \Rightarrow xy' = \dot{z}$$

$$y''(x) = \frac{dy'}{dx} = \frac{d(\dot{z} e^{-t})}{dt} \frac{dt}{dx} = (\ddot{z} e^{-t} - \dot{z} e^{-t}) e^{-t}$$

$$x^2 y'' = \ddot{z} - \dot{z} = (\ddot{z} - \dot{z}) e^{-2t} = (\ddot{z} - \dot{z}) \frac{1}{x^2}$$

4.) $\ddot{z} - \dot{z} - 12 = 0$

$$p(\lambda) = \lambda^2 + \lambda - 12 = 0 \Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{48}{4}}$$

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{7}{2}$$

$$\lambda_1 = 3$$
$$\lambda_2 = -4$$

$$z(t) = c_1 e^{3t} + c_2 e^{-4t}$$

3.) $A = \dfrac{a}{p(m)}$ $\quad a = 1$

$$m = \frac{1}{2}$$

$$p\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{2} - 12$$
$$= -\frac{45}{4}$$

$$z_p(t) = -\frac{45}{4} e^{\frac{t}{2}}$$

$$z(t) = c_1 e^{3t} + c_2 e^{-4t} - \frac{45}{4} e^{\frac{t}{2}}$$

$$y(x) = c_1 x^3 + c_2 x^{-4} - \frac{45}{4} \sqrt{x}$$

Beispiel

$$2yy'' - 3(y')^2 = 4y^2$$ nicht lineare Dgl.

$F(x, y, y', y'') \Rightarrow x$ fehlt

x fehlt: $F(y, y', y'') = 0 \Rightarrow$ y ... unabhängige Variable

$y' = p(y)$ abhängige Variable

$$y'' = \frac{dy'}{dx} = \frac{dp(y)}{dx}$$

$$= \frac{dp}{dy} \cdot \frac{dy}{dx} = p' \cdot p$$

$$2yp' \cdot p - 3p^2 = 4y^2 \quad \Big| \cdot \frac{1}{2py}$$

$$p' - \frac{3}{2}\frac{p}{y} = 2\frac{y}{p} \quad \Big|$$ subst: $\frac{p}{y} = z$

$\Rightarrow \quad p = z \cdot y$

$\Rightarrow \quad p' = z'y + z$

$$z'y + \underbrace{z - \frac{3}{2}z}_{-\frac{1}{2}z} = 2 \cdot \frac{1}{z} \quad \Rightarrow \quad z'y = \frac{z}{2} + \frac{2}{z}$$

$$\int \frac{dz}{\frac{z}{2} + \frac{2}{z}} = \int \frac{dy}{y} \Rightarrow \int \frac{2z\,dz}{z^2 + y} + \ln|C_1|$$

$$\ln|y| = \ln|z^2+4| + \ln|c_1|$$

$$\Rightarrow y = c_1(z^2+4)$$

$$= c_1\left(\frac{p^2}{y^2}+4\right)$$

$$\left(\frac{y}{c_1}-4\right)y^2 = p^2$$

$$\Rightarrow (y')^2 = \quad y' = \sqrt{\frac{y}{c_1}-4}$$

Lösen des AWP: $\quad y(0)=1$, $\quad y'(1)=0$

$$0 = \sqrt{\frac{1}{c_1}-4} \Rightarrow c_1 = \frac{1}{4}$$

$$0 = 0 + c_2$$

$$\sqrt{y-1} = \tan(x) \Rightarrow y = \tan^2(x)+1$$

$$\Rightarrow \int \frac{dy}{2y\sqrt{y-1}} = \underbrace{\int dx}_{x} + c_2$$

$$\frac{1}{2}\int \frac{dy}{y\sqrt{y-1}}$$

$y-1 = v^2$

$y = v^2+1$

$dy = 2v\,dv$

$$\frac{1}{\cancel{2}}\int \frac{\cancel{2}\cancel{v}\,dv}{(v^2+1)\cancel{v}} = \int \frac{dv}{v^2+1} = \arctan(v) = \arctan(y\sqrt{y-1})$$