Dgl Rep.

Note Title 29.04.2008

Reduktion der Ordnung:

$$y'' + a(x)y' + b(x)y = 0$$

allgemeine Lösung: $y = c_1 y_1(x) + c_2 y_2(x)$

$y_1(x), y_2(x)$ linear unabhängig

AWP: $y(x_0) = y_0$

$y'(x_0) = \bar{y}_0$

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y_0$$
$$c_2 y_1'(x_0) + c_2 y_2'(x_0) = \bar{y}_0$$

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ \bar{y}_0 \end{pmatrix}$$

inhomogenes lin. GLS

$\to$ Wronski Det $\neq 0$, wenn $y_1(x), y_2(x)$ linear unabhängig

$y_1(x)$ sei bekannte Lösung $\neq 0$

$$\frac{y_2(x)}{y_1(x)} \neq \text{const.}$$

$$y_2(x) = v(x)\, y_1(x)$$

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2 v' y_1' + v y_2''$$

$$v''y_1 + 2v'y_1' + vy_1'' + Q(x)(v'y_1 + vy_1') + B(x)\, v y_1 = 0$$

$$v\underbrace{[y_1'' + Q(x)y_1' + B(x)y_1]}_{=0} + v''y_1 + v'[2y_1' + Q(x)y_1] = 0$$

$$v' = z$$

$$\Rightarrow z'y_1 + z[2y_1' + Q(x)y_1] = 0$$ lineare Dgl 1. Ordnung

Beispiel 2.)

$$x^4y'' + 2x^3y' - y = 0$$

$$y_1(x) = e^{-x}$$

$$y_1' = e^{-x}\left(-\frac{1}{x^2}\right)$$

$$y_1'' = e^{-x}\left(\frac{1}{x^4} - \frac{1}{x^2}\right)$$

$$\Big[\underbrace{x^4\left(\frac{1}{x^4} + 2\frac{1}{x^3}\right)}_{1+2x} + \underbrace{2x^3\left(-\frac{1}{x^2}\right)}_{-2x}\underbrace{-1}_{-1}\Big]e^{-x} = 0 \quad = 0 \checkmark$$

$$y'' + \frac{2}{x}y' - \frac{1}{x^4}y = 0 \quad \Big| \quad 2y_1' + Q(x)y_1 = 2e^{-x}\left(-\frac{1}{x^2}\right) + \frac{2}{x}e^{-x}$$

$$= e^{-x}\left[-\frac{2}{x^2} + \frac{2}{x}\right]$$

$$z'\cancel{e^{-x}} + 2z\cancel{e^{-x}}\left[\frac{1}{x} - \frac{1}{x^2}\right] = 0$$

$$\frac{dz}{z} = 2\left[\frac{1}{x^2} - \frac{1}{x}\right]dx \Rightarrow \ln|z| = -2x^{-1} - 2\ln|x| + \ln C$$

$$z = \frac{C_2}{x^2}e^{-\frac{2}{x}} = \frac{dv}{dx}$$

$$\vartheta(x) = c\int \frac{e^{-\frac{2}{x}}}{x^2}\,dx = \underbrace{\frac{c}{2}}_{\stackrel{!}{=}1 \text{ gewählt}} e^{-\frac{2}{x}}$$

$$y_2(x) = e^{\frac{2}{x}} e^{-x} = e^{-x}$$

$$\Rightarrow \quad y(x) = c_1 e^{-x} + c_2 e^{-x}$$

Beispiel 3.)

$$(1) \quad xy'' + (x-1)y' + (3-12x)y = 0$$

$$\left.\begin{aligned} y_1(x) &= a \cdot e^{bx} \\ y_1' &= a\,b\,e^{bx} \\ y_1'' &= a\,b^2 e^{bx} \end{aligned}\right\} \text{Lösung?}$$

$$\downarrow$$

$$e^{bx}\left[x\,\cancel{a}\,b^2 + (x-1)\cancel{a}\,b + (3-12x)\cancel{a}\right] = 0$$

$$x\underbrace{(b^2+b-12)}_{=0} \underbrace{-b+3}_{b=3} = 0$$

a frei wählbar

$\Rightarrow a = 1$ gewählt

$$y_1 = e^{3x}$$
$$y_1' = 3e^{3x}$$
$$y_1'' = 9e^{3x}$$

$$\Rightarrow (1) \quad y'' + \underbrace{\left(1-\frac{1}{x}\right)}_{a(x)} y' + \left(\frac{3}{x} - 12\right)y = 0$$

$$2y_1' + a(x)\,y_1 = 6e^{3x} + \left(1-\frac{1}{x}\right)e^{3x} = e^{3x}\left[7-\frac{1}{x}\right]$$

$$z'\,e^{3x} + z\,e^{3x}\left[7-\frac{1}{x}\right] = 0$$

$$\Rightarrow \frac{dz}{z} = \left[\frac{1}{x} - 7\right]dx \Rightarrow \ln|z| = \ln|x| - 7x + \ln C$$

$$z = \underset{\downarrow\ =1\text{ gewählt}}{C}\cdot x\cdot e^{-7x} = v' = \frac{dv}{dx}$$

$$v(x) = \int \underset{u}{x}\,\underset{v'}{e^{-7x}}\,dx = -\frac{1}{7}e^{-7x} + \frac{1}{7}\underbrace{\int e^{-7x}\,dx}_{\frac{1}{7}e^{-7x}}$$

$$= -\frac{1}{7}e^{-7x} + \frac{1}{49}e^{-7x}$$

$$= -\frac{1}{7}e^{-7x}\left[x + \frac{1}{7}\right]$$

$$y_2(x) = -\frac{1}{7}e^{-4x}\left[x + \frac{1}{7}\right]$$

$$y_1(x) = e^{3x}$$

Beispiel 4.)

$$xy'' - y' + 4x^3 y = 0 \qquad x > 0$$

$$y_1(x) = \sin(x^2) \qquad y_2(x) = ?$$

$$y_1'(x) = 2x\cos(x^2)$$

$$y_1''(x) = -4x^2 \sin(x^2)$$

$$z' + \left(4x \cdot \frac{\cos(x^2)}{\sin(x^2)} - \frac{1}{x}\right) z = 0$$

$$\int \frac{dz}{z} = \int\left(\frac{1}{x} - 4x\,\frac{\cos(x^2)}{\sin(x^2)}\right) dx$$

$$2\int x\,\frac{\cos(x^2)}{\sin(x^2)}\,dx = \ln|\sin(x^2)|$$

$$\sin(x^2) = u$$
$$\cos(x^2)\,dx = du$$

$$\ln|z| = \ln|x| + \ln|c| - 2\ln|\sin(x^2)|$$

$$z = Cx \cdot \sin^{-2} x^2 = v' = \frac{dv}{dx}$$

$$v(x) = \int \frac{x}{\sin^2(x^2)}\,dx \quad / \; x^2 = w, \; 2x\,dx = dw$$

$$= \frac{1}{2}\int \frac{dw}{\sin^2(w)}$$

$$= -\frac{1}{2}\cot(x^2)$$

$$y(x) = c_1 \sin(x^2) + c_2\cos(x^2)$$