Dgl - Rep

Note Title 08.04.2008

Berechnung orthogonaler Trajektorien der Kurvenschar:

1 $\quad x^3 = (2C - x)y^2 \qquad C \ldots$ Scharparameter

$$x^3 y^{-2} = 2c - x$$

$$3x^2 y^{-2} + x^3(-2)y^{-3}y' = -1$$

$$2x^3 y^{-3} y' = 1 + 3x^2 y^{-2}$$

2 $\quad y' = \dfrac{1 + 3x^2 y^{-2}}{2x^3 y^{-3}}$ Dgl. für Kurvenschar

3 $\quad y' = -\dfrac{1}{y'}$ für orthogonale Trajektorien

$$y' = -\frac{2X^3 Y^{-3}}{1 + 3X^2 Y^{-2}}$$

4 $\quad y' = -\dfrac{2 \cdot \left(\frac{X}{Y}\right)^3}{1 + 3\left(\frac{X}{Y}\right)^2} \quad / \cdot \dfrac{\left(\frac{X}{Y}\right)^{-3}}{\left(\frac{X}{Y}\right)^{-3}}$

$$= -\frac{2}{\left(\frac{x}{y}\right)^{-3} + 3\left(\frac{x}{y}\right)^{-1}}$$

$$= -\frac{2}{\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)}$$

⑤ subst: $\frac{y}{x} = z \Rightarrow y = z \cdot x$

$y' = z' \cdot x + z$

$$y' = z' x + z = \frac{-2}{z^3 + 3z}$$

$$z' \cdot x = \frac{-2 - z(z^3 + 3z)}{z^3 + 3z}$$

$$= \frac{3z + z^3}{\underbrace{z^4 + 3z^2 + 2}_{(z^2+2)(z^2+1)}} \, dz = -\frac{dx}{x}$$

⑥ Partialbruch:

$$\frac{3z + z^3}{(z^2+2)(z^2+1)} = \frac{A + Bz}{z^2+2} + \frac{C + Dz}{z^2+1}$$

$$3z + z^3 = (A + Bz)(z^2+1) + (C + Dz)(z^2+2)$$

$$= z^3 \underbrace{(B+D)}_{\stackrel{!}{=}1} + z^2 \underbrace{(A+C)}_{\stackrel{!}{=}0 \Rightarrow A = C = 0} + z \overbrace{(B+2D)}^{\stackrel{!}{=}3} + \underbrace{A + 2C}_{=0}$$

$$\Rightarrow A = C = 0$$

$$D = 2$$

$$B = -1$$

⑦ $$\int \frac{-z}{z^2+2} dz + 2 \int \frac{z}{z^2+1} dz = -\frac{dx}{x}$$

$$-\frac{1}{2} \ln(z^2+2) + \ln(z^2+1) = -\ln(x) + \ln(c)$$

$$\frac{z^2+1}{\sqrt{z^2+2}} = \frac{c}{x} = \frac{\left(\frac{y}{x}\right)^2+1}{\sqrt{\left(\frac{y}{x}\right)^2+2}} = \frac{\frac{1}{x^2}(y^2+x^2)}{\frac{1}{x}\sqrt{y^2+2x^2}}$$

8 $$y^2+x^2 = c\sqrt{y^2+2x^2}$$

Beispiel 2:

1 $$(\underbrace{2xy+y^3}_{M})\,dx + (\underbrace{3x^2+xy^2}_{N})\,dy = 0$$

~ für Homogen müssten alle Potenzzahlen gleich sein;

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{exakt}: M_y = N_x$$

$$2x+3y^2 \neq 6x+y^2$$

$$N_x - M_y = 6x+y^2-2x-3y^2 = 4x-2y^2$$

2 1.) $\mu = \mu(x)$ bzw. $\mu = \mu(y)$

$$\mu(M_y - N_x) = \mu_x M$$

$$f(x) = \frac{M_y - N_x}{M} = \frac{\mu_x}{\mu} \Rightarrow \text{geht nicht}$$

2.) $\mu = X(x)\,Y(y) = x^m y^n$: $\frac{\mu_x}{\mu} = \frac{m x^{m-1} y^n}{x^m y^n} = \frac{m}{x}$

$$N_x - M_y = \frac{\mu_y}{\mu} M - \frac{\mu_x}{\mu} \cdot N$$

$$= \frac{n}{x}(2xy+y^3) - \frac{m}{y}$$

$$4x-2y^2 = n(2x+y^2)-m(3x+y^2)$$
$$= x(2n-3m)+y^2(n-m)$$

$$\begin{aligned} 2n-3m &= 4 \\ n-m &= -2 \\ \hline -m &= 8 \end{aligned} \Rightarrow \begin{aligned} m &= -8 \\ n &= -10 \end{aligned}$$

$$\Rightarrow \mu = x^{-8} \cdot y^{-10}$$

(3) $$x^{-8}y^{-10}\left[(2xy+y^3)\,dx+(3x^2+xy^2)\,dy\right]=0$$

$$\underbrace{(2x^{-7}y^{-9}+x^{-8}y^{-7})}_{\bar{M}}\,dx+\underbrace{(3x^{-6}y^{-10}+x^{-7}y^{-8})}_{\bar{N}}\,dy=0$$

$$\bar{N}_x=\bar{M}_y \Rightarrow -8x^{-8}y^{-10}-7x^{-8}y^{-8}=-18x^{-7}y^{-10}-7x^{-8}y^{-8} \checkmark$$

(4) $$\int_{x_0}^{x}\bar{M}(x,y)\,dx+\int\bar{N}(x_0,y)\,dy=C$$

$$\int_{x_0=1}^{x}(2x^{-7}y^{-9}+x^{-8}y^{-7})\,dx+\int(3y^{-10}+y^{-8})\,dy=C$$

$$-\frac{1}{3}x^{-6}y^{-9}\Big|_1^x-\frac{1}{7}x^{-7}y^{-7}\Big|_1^x-\frac{1}{3}y^{-9}-\frac{1}{7}y^{-7}=C$$

$$-\frac{1}{3}x^{-6}y^{-9}+\cancel{\frac{1}{3}y^{-9}}-\frac{1}{7}x^{-7}y^{-7}+\cancel{\frac{1}{7}y^{-7}}-\cancel{\frac{1}{3}y^{-9}}-\cancel{\frac{1}{7}y^{-7}}=C$$

$$7x^{-6}y^{-9}+3x^{-7}y^{-7}=C'$$