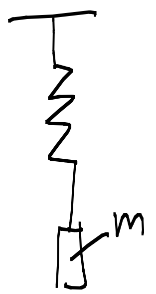
Dgl Vo

Note Title 07.05.2008

Beispiel für homogene Dgl 2. Ordnung



$$\frac{d}{dx}(m v) = \sum F(x) \qquad m\ddot{x} = -k x$$

$$\ddot{x} + \underbrace{\frac{k}{m}}_{\omega^2} x = 0$$

$$x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0 = v_0$$

$$p(\lambda) = \lambda^2 + \omega^2 = 0$$

$$\lambda_{1,2} = \pm i\omega$$

$$\Rightarrow \quad x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$x(0) = c_1 = x_0$$

$$\dot{x}(t) = -\omega c_1 \sin(\omega t) + c_2 \omega \sin(\omega t)$$

$$\dot{x}(0) = \omega_2 c_2 = v_0$$

$$x(t_0) = x_0$$
$$\dot{x}(t_0) = \dot{x}_0$$

$$\Rightarrow \quad x(t) = x_0 \cos(\omega(\underbrace{t-t_0}_{\tilde{t}})) + \frac{\dot{x}_0}{\omega} \sin(\omega(\underbrace{t-t_0}_{\tilde{t}}))$$

selbes Ergebnis: Inversion bezüglich Translation

gilt nur für autonome Systeme

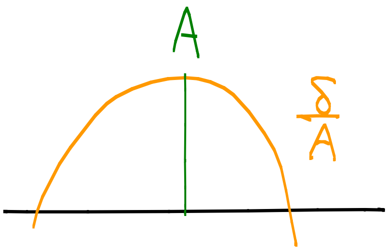
$$\dot{y} = f(y,t) = \dot{y} = f(y)$$

$$y'' + a(x)y' + b(x)y = f(x)$$

$$x_0 = A\cos(\delta)$$

$$\frac{\dot{x}_0}{\omega} = A\sin(\delta)$$

$$A^2\sin^2(\delta) + A^2\cos^2(\delta) = A^2 = x_0^2 + \frac{\dot{x}_0^2}{\omega^2}$$



$$A = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega^2}}$$

$$\tan\delta = \frac{\dot{x}_0}{\omega x_0} \Rightarrow \delta = \operatorname{atan}\left(\frac{\dot{x}_0}{\omega x_0}\right)$$

$$x(t) = A\cos(\delta)\cos(\omega t) + A\sin(\delta)\sin(\omega t)$$
$$= A\cos(\omega t - \delta)$$

betrachten mit Reibung:

$$\ddot{x} + \alpha\dot{x} + \omega^2 x = 0$$

$$p(\lambda) = \lambda^2 + \alpha\lambda + \omega^2 = 0 \Rightarrow \lambda_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \omega^2}$$

$\frac{\alpha^2}{4} > \omega^2$ ... starke Dämpfung

$\frac{\alpha^2}{4} < \omega^2$ ... schwache Dämpfung

$\frac{\alpha^2}{4} = \omega^2$ ... aperiodische

Lineare Dgl n. Ordnung

$$y^{(n)} + a_{n-1} y^{(n-1)}(x) + \dots + a_0 y(x) = 0$$

- $y = e^{\lambda x}$ eine Lösung $\Rightarrow e^{\lambda x}\underbrace{(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0)}=0$

$$p(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\dots(\lambda-\lambda_r)^{m_r} = 0$$

$$m_1 + m_2 + \dots + m_r = 0$$

$$\Rightarrow \quad e^{\lambda_1 x},\ x e^{\lambda_1 x}, \dots, x^{m_1-1} e^{\lambda_1 x}$$
$$e^{\lambda_2 x},\ x e^{\lambda_2}, \dots, x^{m_2-1} e^{\lambda_2 x}$$
$$\vdots$$
$$e^{\lambda_r x},\ x e^{\lambda_r}, \dots, x^{m_r-1} e^{\lambda_r x}$$

Beispiel:

$$y^{IV} + 3y''' + 15y'' + 13y' + 4y = 0$$

$$p(\lambda) = \lambda^4 + 3\lambda^3 + 15\lambda^2 + 13\lambda + 4 = 0$$
$$= (\lambda+1)^3(\lambda+4)^1 \Rightarrow \left.\begin{array}{ll} \lambda_1 = -1 & m_1 = 3 \\ \lambda_2 = -4 & m_2 = 1 \end{array}\right\} \sum m_i \overset{!}{=} n$$

Lösung: $e^{-x},\ x e^{-x},\ x^2 e^{-x},\ e^{-4x}$

$$W(y_1,\dots,y_n) = \begin{vmatrix} e^{-x} & x e^{-x} & x^2 e^{-x} & e^{-4x} \\ -e^{-x} & e^{-x} - x e^{-x} & 2x e^{-x} + x^2 e^{-x} & -4e^{-4x} \\ e^{-x} & \vdots & \vdots & \vdots \\ -e^{-x} & & & \end{vmatrix} = 0$$

$y'' + ay' + by = f(x)$ analog für n. Ordnung

allgemeine Lösung: $y = y_h + y_p$

y_h: allgemeine Lsg der homogenen Gleichung

y_p: allgemeine Lsg der partikulären Gleichung

Überlagerungsprinzip: (Liniarität)

$$f(x) = \sum_{i=1}^{r} c_i f_i(x)$$

$$\underbrace{y'' + ay' + by}_{y_{p_i}} = f_i(x) \quad i = 1, \dots, r$$

$$y_p(x) = \sum_{i=1}^{r} c_i \cdot y_{p_i}(x)$$

Beispiel:

$$y'' + 4y = \underbrace{\sin(x)}_{f_1(x)} + \underbrace{x^2}_{f_2(x)}$$

i.) $y'' + 4y = \sin(x)$

ii.) $y'' + 4y = x^2$

H.) $y'' + 4y = 0 \Rightarrow p(\lambda) = \lambda^2 + 4\lambda = 0 \Rightarrow \lambda_{1,2} = \pm 2i$

$y_h = c_1 \cos(2x) + c_2 \sin(2x)$

Ansatz: ii.) $y'' + 4y = x^2$

$$y_{p\,Ansatz} = a_0 + a_1 x + a_2 x^2$$
$$y_p' = a_1 + 2a_2 x$$
$$y_p'' = 2a_2$$

$$2a_2 + 4(a_0 + a_1 x + a_2 x^2) = x^2$$ Ansatz muss Gl entsprechen

$$\left.\begin{array}{llll} x^0: & 2a_2 + 4a_0 & = 0 & \Rightarrow a_0 = -\frac{1}{8} \\ x^1: & 4a_1 & = 0 & \Rightarrow a_1 = 0 \\ x^2: & 4a_2 & = 1 & \Rightarrow a_2 = \frac{1}{4} \end{array}\right\} y_p = -\frac{1}{8} + \frac{1}{4}x^2$$

Ansatz: i.) $y'' + 4y = \sin(x)$

$$y_{p\,Ansatz} := a\sin(x) + b\cos(x)$$
$$y_p' = a\cos(x) - b\sin(x)$$
$$y_p'' = -a\sin(x) - b\cos(x)$$

$$-a\sin(x) - b\cos(x) + 4(a\sin(x) + b\cos(x))$$

$$\left.\begin{array}{llll} \sin(x) & -a + 4a & = 1 & \Rightarrow a = \frac{1}{3} \\ \cos(x) & -b + 4b & = 0 & \Rightarrow b = 0 \end{array}\right\} y_p = \frac{1}{3}\sin(x)$$

$$y_{allgemein}: c_1\cos(2x) + c_2\sin(2x) - \frac{1}{8} + \frac{1}{4}x^2 + \frac{1}{3}\sin(x)$$

alternativ über anderen Ansatz:

$$y_p(x): a_0 + a_1 x + a_2 x^2 + a_3\sin(x) + a_4\cos(x)$$