Analysis Übung 9:

Note Title 31.05.2007

40.) $u_{tt} = c^2 u_{xx} \qquad 0 \le x \le L$

$u(0,t) = 0 \qquad u(x,0) = x(L-x) = f(x)$

$u(L,t) = 0 \qquad u_t(x,0) = 0 \qquad = g(x)$

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}x\right) \cdot \left[c_k \sin\left(\frac{k\pi t}{L}c\right) + d_k \cos\left(\frac{k\pi t}{L}\right)\right]$$

$$u(x,0) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right) d_k \overset{!}{=} x(L-x)$$

$$u_t(x,0) = \sum_{k=1}^{\infty} c_k \cdot \frac{k\pi}{L} c \cdot \sin\left(\frac{k\pi}{L}x\right) \overset{!}{=} 0$$

Setze:

$f(-x) = -f(x) \qquad$ für $0 \le x \le L$

$g(-x) = -g(x) \qquad$ für $0 \le x \le L$

$$c_k = \frac{2}{L}\int_0^L 0 \;\ldots\; = 0$$

$$d_k = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{k\pi}{L}x\right) dx$$

$$= \frac{2}{L}\int_0^L x \cdot (L-x) \cdot \sin\left(\frac{k\pi}{L}x\right) dx$$

$$= \int \underbrace{(xL - x^2)}_{u} \cdot \underbrace{\sin\left(\frac{k\pi x}{L}\right)}_{v'}$$

$$u' = L - 2x \qquad v = -\frac{\cos\left(\frac{k\pi x}{L}\right)\cdot L}{k\pi \cdot 1}$$

$$\int u v' \, dx = u \cdot v - \int v u' \, dx$$

$$= -(xL - x^2) \cdot \frac{\cos\left(\frac{k\pi x}{L}\right)\cdot L}{k\pi} - \int \underbrace{-\frac{\cos\left(\frac{k\pi x}{L}\right)\cdot L}{k\pi}}_{v'} \cdot \underbrace{(L - 2x)}_{u}$$

$$v = \frac{\sin\left(\frac{k\pi x}{L}\right)\cdot L}{k\pi \cdot 1} \qquad u' = -2$$

$$-(xL - x^2) \cdot \frac{\cos\left(\frac{k\pi x}{L}\right)\cdot L}{k\pi} - \left[\left(\frac{L}{k\pi}\right)^2 (L - 2x)\left(\frac{\sin\left(\frac{k\pi x}{L}\right)}{k\pi}\right)\right] - \int \frac{\sin\left(\frac{k\pi x}{L}\right)\cdot L}{k\pi} (-2)$$

$$= \frac{-xL^2 + x^2 L}{k\pi} \cdot \cos\left(\frac{k\pi x}{L}\right) - \frac{(L - 2x)L^2}{(k\pi)^2} \sin\left(\frac{k\pi x}{L}\right) + \left(\frac{L}{k\pi}\right)^3 \cos\left(\frac{k\pi}{L}x\right)$$

$$d_k = \frac{2}{L} \cdot \left[\frac{Lx^2 - Lx}{k\pi} \cos\left(\frac{k\pi x}{L}\right) + \left(\frac{L}{k\pi}\right)^2 (L - 2x) \sin\left(\frac{k\pi x}{L}\right) - \right.$$
$$\left. - 2\left(\frac{L}{4\pi}\right)^3 \cos\left(\frac{k\pi}{L}x\right) \right|_0^L$$

$$= \frac{2(L^2 - L)}{k\pi} (-1)^4 + \frac{4L^2}{k^3 \pi^3} \left(1 - (-1)^k\right) =$$

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right) \cdot \left(\frac{2L^2 - 2L}{k\pi}(-1)^k + \frac{4L^2}{k^3\pi^3}\left(1 - (-1)^k\right)\right) \cdot \cos\frac{k\pi t}{L}$$

41.) $u_{tt} = 4 \cdot u_{xx}$

$$u(0,t) = 0 \qquad u(x,0) = x(L-x)$$
$$u(L,t) = 0 \qquad u_t(x,0) = \frac{L}{2} - \left|x - \frac{L}{2}\right|$$

d_k aus Bsp. 40.)

$$c_k = \frac{2}{L} \cdot \int_0^L \frac{L}{2} - \left|x - \frac{L}{2}\right| \cdot \sin\left(\frac{k\pi x}{L}\right) dx$$

$$\Rightarrow \int_0^{L/2} \left(\frac{L}{2} - \left|\frac{L}{2} - x\right|\right) \dots + \int_{\frac{L}{2}}^{L} \left(\frac{L}{2} - \left(x - \frac{L}{2}\right)\right) \dots$$

$$= \frac{2}{L} \int_0^{L/2} \underbrace{(x)}_{u_1} \underbrace{\sin\left(\frac{k\pi x}{L}\right)}_{v'} dx + \frac{2}{L} \int_{\frac{L}{2}}^{L} \underbrace{(L-x)}_{u} \underbrace{\sin\left(\frac{k\pi x}{L}\right)}_{v'} dx$$

$$u_1' = 1 \qquad v = -\frac{\cos\left(\frac{k\pi x}{L}\right) L}{k\pi \cdot 1}$$

$$u_2' = -1$$

$$\left(x \, \frac{\cos\left(\frac{k\pi x}{L}\right) L}{k\pi} - \int_0^{L/2} 1 \cdot \frac{L}{k\pi} \cos\left(\frac{k\pi x}{L}\right)\right.$$

$$+ \left((L-x) \cdot \frac{L}{k\pi} \cdot \left(-\cos \frac{k\pi x}{L}\right)\right) - \int_{L/2}^{L} (L-x) \cdot \frac{L}{k\pi} \cdot \cos\left(\frac{k\pi x}{L}\right)$$

$$= \frac{2}{L}\Big[\frac{xL}{k\pi}\cos\left(\frac{k\pi x}{L}\right) - \left(\frac{L}{k\pi}\right)^2 \sin\left(\frac{k\pi x}{L}\right)\Big|_0^{\frac{L}{2}} +$$

$$\frac{L(L-x)}{k\pi}\cdot\cos\left(\frac{k\pi x}{L}\right) - \left(\frac{L}{k\pi}\right)^2 \sin\left(\frac{k\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\Big]$$

$$= -\frac{2L}{k^2\pi^2}\cdot\sin\left(\frac{k\pi}{L}\right)\cdot 2$$

$$= -\frac{4L}{k^2\pi^2}\cdot\sin\left(\frac{k\pi}{L}\right)$$

$$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right)\Big[\Big(\frac{2L^2-2L}{k\pi}(-1)^k + \frac{4L^2}{k^3\pi^3}\left(1-(-1)^k\right)$$

$$\cos\left(\frac{4k\pi t}{L}\right)\Big) - \frac{L^2}{k^2\pi^2}\sin\left(\frac{4k\pi t}{L}\right)\Big]$$

42.) $\sum_{k=1}^{n} \cos(2k-1)x$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$= \frac{x}{2}\sum_{k=1}^{n}\left(e^{i(2k-1)} + e^{-i(2k-1)}\right)$$

$$= \frac{x}{2}e^{-i}\sum_{k=1}^{n} e^{i2k} + \frac{x}{2}e^{i}\sum_{k=1}^{n} e^{-i2k}$$

$$= \frac{x}{2}e^{-i}\sum_{k=0}^{n} e^{i2k} - 1 + \frac{x}{2}e^{i}\cdot\sum_{k=0}^{n} e^{-i2k} - 1$$

$$= \frac{x}{2}e^{-i}\cdot\frac{e^{i2(n+1)}-1}{e^{i2}-1} + \frac{x}{2}e^{i}\,\frac{e^{-i2(n+1)}-1}{e^{-i2}-1}$$

$\Longrightarrow$

$\sum_{k=1}^{n} \sin(2k-1)x$

$$= \frac{x}{2}\sum_{k=1}^{n}\left(e^{+i(2k-1)} - e^{-i(2k-1)}\right)$$

$$= \frac{x}{2}\sum_{k=0}^{n}\left(e^{i(2k-1)} - e^{-i(2k-1)}\right) - 0$$

$$= \frac{x}{2}e^{-i}\sum_{k=0}^{n} e^{i(2k)} - \frac{x}{2}e^{i}\sum_{k=0}^{n} e^{-i2k}$$

$$= \frac{x}{2}e^{-i}\,\frac{e^{i2(n+1)}-1}{e^{i2}-1} - \frac{x}{2}e^{i}\,\frac{e^{-i2(n+1)}-1}{e^{-i2}-1}$$

43.) $\sum_{n=0}^{\infty} \frac{n^n}{n!} z^n$

$$R = \lim_{n\to\infty} \frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{(n+1)} \cdot \cancel{n!}}{n^n \cdot (n+1)\cancel{!}}$$

$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \frac{\cancel{(n+1)^1} (n+1)^n}{n^n \cancel{(n+1)}}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= e^{\wedge}\left(n \cdot \ln\left(\frac{n+1}{n}\right)\right)$$

$$= e^{\wedge}\left(n \cdot \ln\left(1 + \frac{1}{n}\right)\right)$$

Hospital

$$= e^{\wedge}\left(\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right)$$

$$= e^{\wedge}\left(\frac{\frac{1}{1+\frac{1}{n}} (-1) \cancel{\frac{1}{n^2}}}{-\cancel{\frac{1}{n^2}}}\right)$$

$$R = \lim_{n\to\infty} e^{\wedge(-1)} = \frac{1}{e}$$

$\sum_{n=0}^{\infty} z^{n^2}$

$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$ / $a_n = 1$

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{1}} = \frac{1}{1} = 1$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} z^n$$

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$R = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2}$$

$$= \frac{n^2}{n^2+2n+1}$$

$$= \frac{\cancel{n^2} \cdot 1}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}$$

$$R \lim_{n\to\infty} = \frac{1}{1+0+0} = 1$$